Nepal Algebra Project(NAP)
Central Department of Mathematics
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Fields and Galois Theory

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Summary of NAP: Module 1, Lecture 6 - Wednesday May 18 2016

Reviewed the Finite-algebraic Theorem from Lecture 5: For F subseteq E fields, these are equivalent:

1. E/F is finite (i.e. finite vector space dimension).
2. E/F is finitely generated as a field extension and E is algebraic over F.
3. There exist algebraic elements α_1, α_2, $\dots$, α_n of E, such that $E = F(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Stated and proved:

Corollary. *Let K/F be a field extension and let $\overline{F}$ be the set of elements of K that are algebraic over F. Then $\overline{F}$ is a subfield of K with $F \subset \overline{F} \subset K$. The field $\overline{F}$ is called the algebraic closure of $F \in K$.*

Discussed the set of algebraic numbers in the complex numbers $\mathbb{C}$ over the rational numbers $\mathbb{Q}$.
Showed that for $\alpha = \sqrt{2} + \sqrt{3}, \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Later showed that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ and consequently that $\big[\mathbb{Q}(\alpha) : \mathbb{Q}\big] = 4$ (By Eisenstein $x^2 - 2$, and $x^2 - 3$ are irreducible over $\mathbb{Q}$ and so $\sqrt{2}, \sqrt{3} \notin \mathbb{Q}$).
Discussed an example of the Louisville numbers (with big gaps in the decimal expansion): $\sum_{i=1}^{\infty} 1/10^{i!}$. These are transcendental and it is easier to show they are transcendental than other transcendental numbers (see book).
Discussed countable and uncountable cardinalities.
Stated and proved:

Theorem. *For K/F a field extension with F countable, the set of elements of K that are algebraic over F is countable.*

To prove this, stated and outlined/picture-proved N lemmas:
For X, X_n, Y sets

1. X, Y countable $\implies X \times Y$ countable;
2. X countable and an injection takes Y into X $\implies Y$ is countable;
3. X countable and a surjection takes $X \to Y \implies Y$ countable;
4. Y a countable union of countable sets $X_n \implies Y$ is countable;
5. F a countable field and V a vector space over F with a countable basis $\implies V$ is countable;

$\vdots$

N. If X is countable and $Y \to X$ is countable-to-one, then Y is countable.

Corollary: In the complex numbers $\mathbb{C}$, the subfield of elements that are algebraic over the rationals $\mathbb{Q}$ is countable. Elements of this set are called algebraic numbers.
Proved part 1 of **Proposition 1.20** Multiplication of degree (part 2 was proved in Lecture 5).
Proved the following tw statements:

Proposition (algebraic extensions are transitive)**.** *If $F \subseteq K \subseteq L$ are fields with K/F algebraic and L/K algebraic, then L/F is algebraic.*

Proposition. *If $F \subseteq K \subseteq L$ are fields and $L = F[\alpha]$ where alpha is algebraic, then the minimal polynomial for α over K divides the minimal polynomial for α over F.*

Worked two problems:

- Show if $[L : F] = p$, a prime integer, then there are no fields K properly between F and L.
- If $F \subseteq L$ is algebraic and T is an integral domain with $F \subseteq T \subseteq L$, prove that T is a field.

Another problem the students might like to try for extra practice is **Exercise 1-5** in Milne.