Modular Forms: Background and motivation

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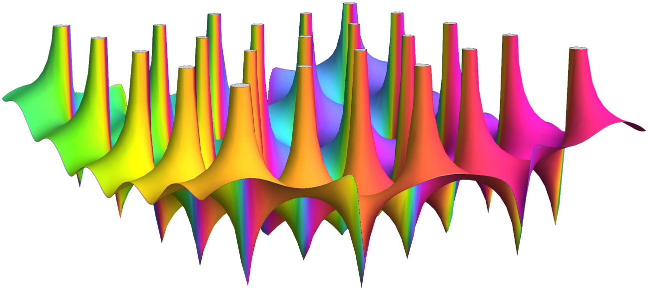

CIMPA research school
Group Actions in Arithmetic and Geometry
Gadjah Mada University, Yogyakarta, Indonesia
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Holomorphic functions

Definition

A function $f : \Omega \to \mathbb{C}$ is *complex differentiable* at $z_0 \in \Omega$ if and only if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite, in which case is denoted with $f'(z_0)$.

Holomorphic functions

Cauchy-Riemann equations

For a function

$$f : \Omega \to \mathbb{C}, \quad \Omega \subseteq \mathbb{C} \textit{ open}, \; z_0 \in \Omega$$

the following statements are equivalent:

(a) f *is complex differentiable at* z_0*.*

(b) f *is totally differentiable at* z_0 *in the sense of real analysis and*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

where $u = \mathrm{Re}(f)$ *and* $v = \mathrm{Im}(f)$*.*

Holomorphic functions

Terminology

A function

$$f : \Omega \to \mathbb{C}, \quad \Omega \subseteq \mathbb{C}, \text{ open}$$

is said to be **holomorphic** *in* Ω *if it is complex differentiable at every point of* D.
f *is said to be* **holomorphic** *at* $z_0 \in \Omega$ *is there exists an open neighborhood* $U \subseteq D$ *of* z_0 *such that* f *is holomorphic in* U.

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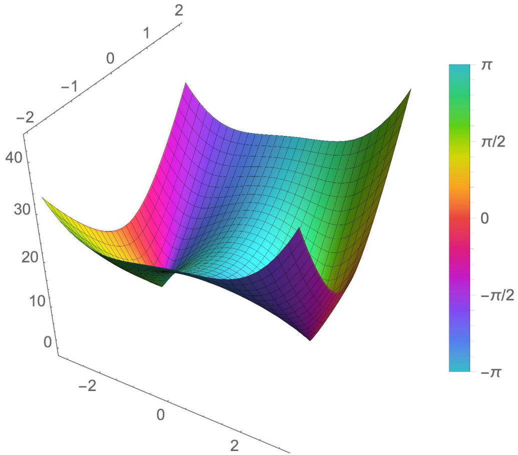
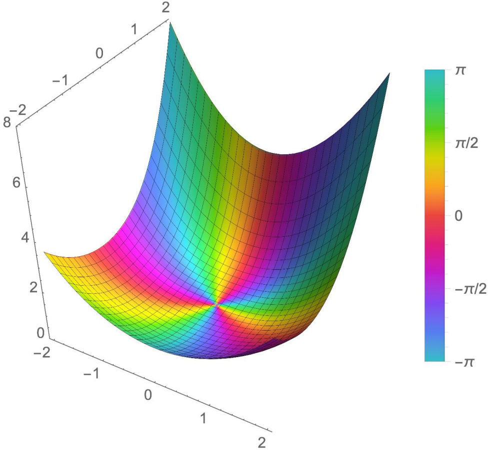
The function $z \mapsto \bar{z}$ is complex differentiable at $z = 0$ but is not holomorphic at $z = 0$, because z_0 is the only point where is complex differentiable.

Visualization



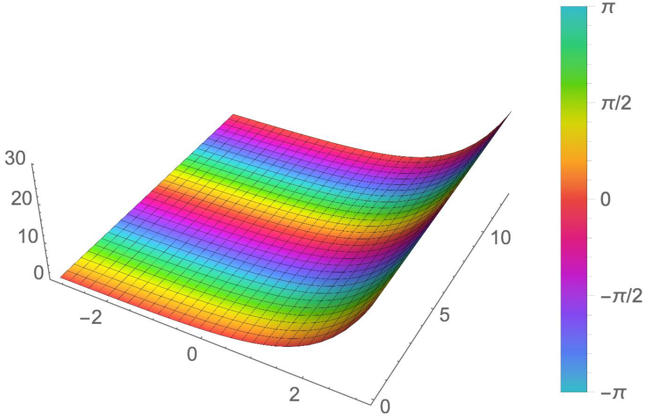
$$f(z) = \frac{1}{7}z^3 + 3z^2 - z - 15$$

Visualization



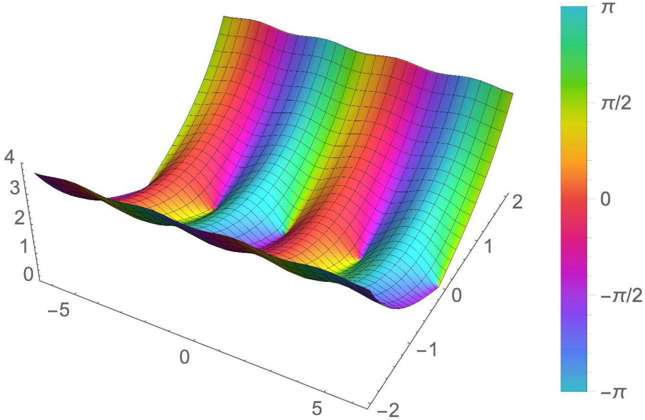
$$f(z) = \frac{z^3}{z-4i}$$

Visualization



$$f(z) = exp(z)$$

Visualization



$$f(z) = sin(z)$$

Complex line integrals

Definition

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuous curve, $f : \Omega \to \mathbb{C}$ be a continuous function, and suppose $\gamma([a, b]) \subseteq \Omega$. Then we define the **line integral of** f **along** γ as

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

Complex line integrals

Definition

By a **domain** we shall mean an arcwise connected open set $D \subseteq \mathbb{C}$.

Theorem

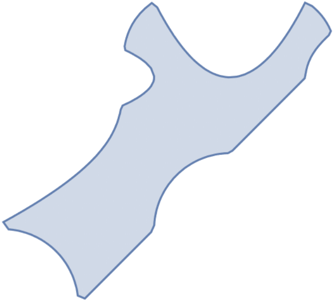
For a continuous function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ a domain, the following are equivalent

(a) *f has a primitive*

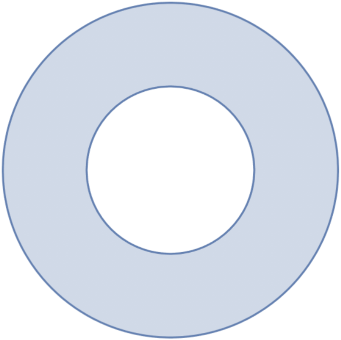
(b) *The integral of f along any closed curve in D vanishes*

(c) *The integral of f over any curve in D depends only on the beginning and end points of the curve*

Domains



simply connected



not simply connected

Cauchy integral formulas

Cauchy Theorem

Let $D \subset \mathbb{C}$ be a simply connected domain, $f : D \to \mathbb{C}$ be an holomorphic function and $\gamma : [a, b] \to \mathbb{C}$ a piecewise continuous closed curve. Then

$$\int_\gamma f(z)dz = 0$$

Cauchy integral formulas

We will denote by $\mathrm{U}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ the open disk centered a z_0 and by $\overline{\mathrm{U}}_r(z_0)$ its closure.

Cauchy Integral Formula

Let $D \subset \mathbb{C}$ be a simply connected domain, $f : D \to \mathbb{C}$ be an holomorphic function in D. Suppose that the closed disk $\overline{\mathrm{U}}_r(z_0)$ lies completely within D and let $\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = z_0 + re^{it}$ (so γ goes once around the boundary of $\overline{\mathrm{U}}_r(z_0)$ counterclokwise). Then for each point $z \in \mathrm{U}_r(z_0)$ we have:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Cauchy integral formulas

Generalized Cauchy Integral Formula

With the assumption and notation of the Cauchy integral formulas we have: Every holomorphic function in D is arbitrarily often complex differentiable, each derivative is again holomorphic. For $n \geq 1$ *and all* $z \in \mathrm{U}_r(z_0)$ *we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta = 0$$

Consequences

A function holomorphic on all of $\mathbb{C}$ is called an **entire** function

Exercises

- *Every bounded entire functions is costant (Liouville's theorem)*
- *Each non constant complex polynomial has a root in* $\mathbb{C}$.

Power series representation

Consider a holomorphic function $f : \Omega \to \mathbb{C}$, with Ω open. Suppose that $\mathrm{U}_r(z_0) \subset \Omega$. Let $\rho < r$ and let $\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = z_0 + \rho e^{it}$. Then for each $z \in \mathrm{U}_\rho(z_0)$, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right) = \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n$$

Power series representation

So $\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}}(z - z_0)^n$, it follows that:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \end{aligned}$$

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Thus the power series representation

$$
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
$$

holds for all $z \in \mathrm{U}_r(z_0)$

Singularities

Given $a \in \mathbb{C}$ we will denote by $\dot{\mathrm{U}}_r(a)$ the punctured disk of radius r centered in a:

$$\dot{\mathrm{U}}_r(a) := \{z \in \mathbb{C} \, : \, 0 < |z-a| < r\}.$$

Definition

Let $f : \Omega \to \mathbb{C}$, Ω open, be an holomorphic function. Suppose $a \notin \Omega$ has the property that there exists $r > 0$ such that $\dot{\mathrm{U}}_r(a) \subseteq \Omega$, then a is called **an isolated singularites** of f.

Classification of isolated singularities

Let $f : \Omega \to \mathbb{C}$, Ω open, be an holomorphic function and a an isolated singularity of f.

- a is called a **removable singularity** if there exists an holomorphic function $\tilde{f} : \Omega \cup \{a\} \to \mathbb{C}$ with $\tilde{f} \Big| \Omega = f$.

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- a is called a **pole** if there exists an integer $m \geq 1$ such that $g(z) = (z - a)^m f(z)$ has a removable singularity at a. The smallest integer k with this property is called the **order** of the pole. If $k = 1$ the pole is called simple.

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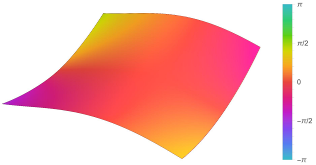
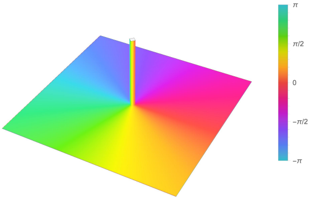
- a is called an **essential singularity** if a is neither removable nor a pole.

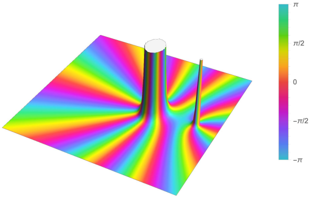
Classification of isolated singularities



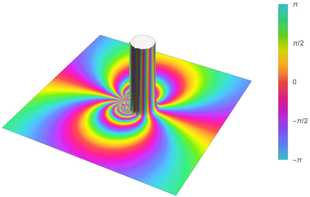
Removable singularity: $f(z) = \frac{\sin(z)}{z}$ around 0.



Simple pole: $f(z) = \frac{1}{z}$, around 0.



Poles of orders 7 and 3: $f(z) = \frac{1}{z^7(z-1)^3}$.



Essential singularity: $f(z) = e^{1/z}$, around 0.

Residues

Let $f : \Omega \to \mathbb{C}$, Ω open, be an holomorphic function and a pole of order $k \geq 1$ for f. Then f can be represented in $\dot{\mathrm{U}}_r(a)$ by a Laurent series

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-a)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$$

The coefficient a_{-1} is called the **residue of f at a** and is denoted by $\mathrm{res}_a(f)$. Note that if a is not singularity of f, then $\mathrm{res}_a(f) = 0$

Meromorphic function

Definition

Let $\Omega \subseteq \mathbb{C}$ be an open set. A **meromorphic function** on Ω is a holomorphic function f on $\Omega \setminus S$, where S is discrete in Ω and each $s \in S$ is a pole for f.

Meromorphic function

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Exercise

Let f be meromorphic in Ω, and $a \in \Omega$. Then

$$\mathrm{res}_a(f'/f) = \mathrm{ord}_a(f)$$

if f is not constantly zero on Ω.

Residues theorem

A closed piecewise smooth curve $\gamma : [a : b] \to \mathbb{C}$ is said to be **simple** if $\gamma(t_1) = \gamma(t_2)$ implies $\{t_1, t_2\} = \{a, b\}$. A piecewise smooth closed simple curve will be called a **contour**. If γ is a contour than γ divides the complex plane in two disconnected parts, one bounded and one unbonded. The bounded one will be called the interior of γ, and will be denoted by I_γ

Theorem

Let $D \subseteq \mathbb{C}$ be a simply connected domain, γ a contour in D, f a meromorphic function in D with only finitely many isolated singularities in the interior of γ. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{z \in I_\gamma} \mathrm{res}_z(f)$$

Computation of residues

Let $D \subset \mathbb{C}$ a domain, a a point in D, and f holomorphic function on $D \setminus \{a\}$, with at most a pole in a, and g holomorphic in D. Then

- If $\mathrm{ord}_a(f) \geq -1$, then $\mathrm{res}_a(f) = \lim_{z \to a}(z-a)f(z)$.
- If $\mathrm{ord}_a(f) = -k < -1$, then

  $$\mathrm{res}_a(f) = \frac{1}{(k-1)!} \lim_{z \to a} \tilde{f}^{(k-1)}(z)$$

  where $\tilde{f}(z) = (z-a)^k f(z)$.
- If $\mathrm{ord}_a(f) > 0$ and $\mathrm{ord}_a(g) = 1$, then $\mathrm{res}_a(f/g) = f(a)/g'(a)$.

Periodic functions

A meromorphic function f on C is said to be periodic, with period ω if

$$f(z+\omega) = f(z) \quad \forall z \in \mathbb{C}$$

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- $\cos(z + 2k\pi) = \cos(z)$. So $\cos(z)$ is periodic with period 2π and all its multiples.

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- $\cos(z + 2k\pi) = \cos(z)$. So $\cos(z)$ is periodic with period 2π and all its multiples.
- $\exp(2\pi i(z+1)) = \exp(z)$. So $\exp(2\pi iz)$ is periodic with period 1 (and all its multiples)

Periodic functions

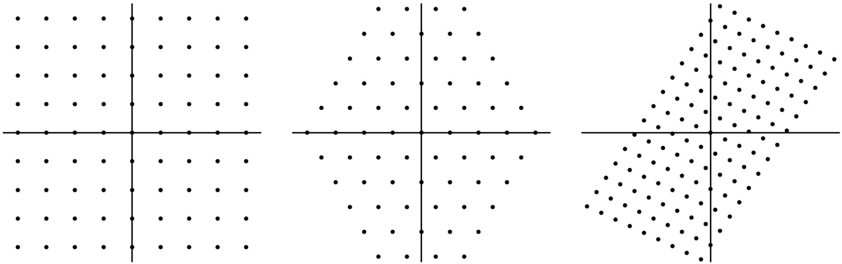
Exercise

Let f be a meromorphic periodic function. Then one of the following holds:

- *f is* **simply periodic**, *i.e. the periods of f are of the form $n\omega_0$, $n \in \mathbb{Z}$.*
- *f is* **doubly periodic**, *i.e. the periods of are of the form $n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbb{Z}$, and ω_1 and ω_2 linearly independent over $\mathbb{R}$.*

Elliptic functions

A doubly periodic meromorphic function is called an **elliptic function**. The set of periods of an elliptic function forms a **lattice**, a discrete subgroup of $\mathbb{C}$ whose basis over $\mathbb{Z}$ generates $\mathbb{C}$ over $\mathbb{R}$.



Lattices in the complex plane

Elliptic functions

Let Λ be a lattice in $\mathbb{C}$ generated by ω_1 and ω_2. Given $c \in \mathbb{C}$, the set

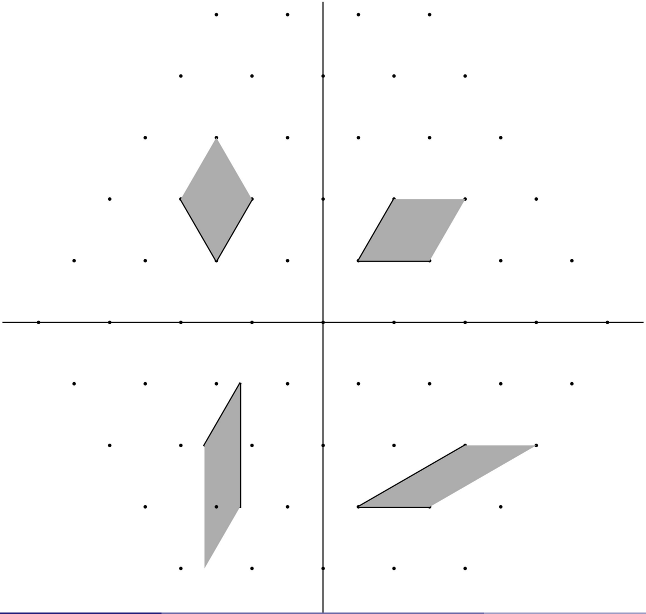
$$\Pi = \{x_1\omega_1 + x_2\omega_2 + c : x_1, x_2 \in \mathbb{R} \text{ and } 0 \leq x_1, x_2 < 1\}$$

Is called a **fundamental parallelogram** for Λ and enjoys the following properties:

- If u_1 and u_2 belong to Π, then $u_1 \not\equiv u_2 \mod \Lambda$.
- If $u \in \mathbb{C}$ then there exits a unique $\bar{u} \in \Pi$ such that $u \equiv \bar{u} \mod \Lambda$.

(Proof: Exercise)

Fundamental domains



Elliptic functions

Let $\Lambda \subset \mathbb{C}$ a lattice. The set of doubly periodic meromorphic functions having Λ as period lattices is denoted by $M(\Lambda)$. Note that $M(\Lambda)$, can be interpreted as the set of meromorphic function on the complex torus $\mathbb{C}/\Lambda$. As a real surfaces such a complex torus looks like:

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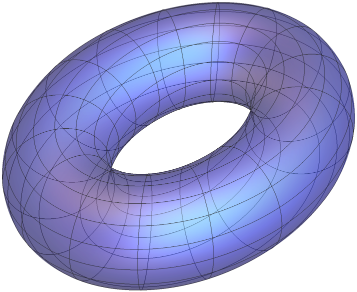


Elliptic functions

Exercises

(a) *An elliptic function must have at least one pole.*

(b) *Let Λ be a lattice and Π a fundamental parallelogram for Λ. Suppose $f, g \in M(\Lambda)$ are such that*

$$\mathrm{ord}_a(f) = \mathrm{ord}_a(g) \quad \textit{for all } a \in \Pi.$$

Then f/g is constant.

Exercises

Let f be an elliptic function with period lattices Λ. Then

- $\sum_{a\in\Pi} \mathrm{res}_a(f) = 0$
- $\sum_{a\in\Pi} \mathrm{ord}_a(f) = 0$
- $\sum_{a\in\Pi} \mathrm{ord}_a(f)a \equiv 0 \mod \Lambda$
- *An elliptic function cannot have only a simple pole in a fundamental domain.*

(Hint: use the Residues theorem) You will need to use the following fact: If $f(a) = f(b)$, and both f and f' do not vanish the line joining a and b, then $\frac{1}{2\pi i}\int_a^b \frac{f'(z)}{f(z)}dz \in \mathbb{Z}$.

Elliptic functions

Let $\Lambda \subset \mathbb{C}$ be a lattice. Consider the following series of meromorphic function:

$$\frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

It can be proven that the series converges normally in any disk $\mathrm{U}_r(0)$ and hence defines a meromorphic function on all of $\mathbb{C}$. The function it converges to is called the **Weiestrass $\wp$-function** and is denoted by:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

By construction $\wp_\Lambda(z)$ is an even function, has a double pole in the origin and in every point of the lattice Λ and it belongs to $M(\Lambda)$.

Elliptic functions

The derivativative of $\wp_\Lambda(z)$ is also an elliptic function for the lattice Λ.

$$\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

and it has a triple pole at the points of the lattice Λ and no other singularities.

Exercise (Structure theorem for $M(\Lambda)$)

Let $\Lambda \in \mathbb{C}$ be a lattice. Every $f(z)$ can be written in the following form

$$f(z) = R_1(\wp_\Lambda(z)) + \wp'_\Lambda(z) R_2(\wp_\Lambda(z))$$

where R_1 and R_2 rational functions.

Eisenstein series

Let $\Lambda \subset \mathbb{C}$ be a lattice. Given $n \geq 3$ the series

$$G_n(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-n}$$

converges absolutely and it gives the Laurent series for both $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$, namely

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2}(\Lambda) z^{2n}$$

and

$$\wp'_\Lambda(z) = -\frac{2}{z^3} + \sum_{n=1}^{\infty} 2n(2n+1) G_{2n+2}(\Lambda) z^{2n-1}$$

Theorem (Algebraic differential equation for $\wp_\Lambda$)

Set $g_2(\Lambda) = 60\,G_4(\Lambda)$ *and* $g_4(\Lambda) = 160\,G_6(\Lambda)$. *Then*

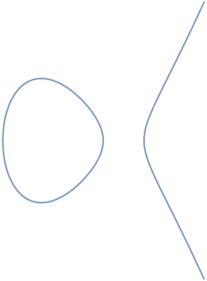
$$\wp_\Lambda'(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda)$$

Elliptic curves

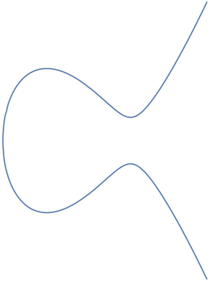
Thus the image of $\mathbb{C}/\Lambda$ via the function

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2(\mathbb{C}) \\ [z]_\Lambda &\longmapsto [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1] \end{aligned}$$

admits a description as a cubic curve in the complex projective plane of the form $y^2 = x^3 + Ax + B$. Such curves are elliptic curves. The real locus of such a curve looks like:



$y^2 = x^3 - 10x + 8$



$y^2 = x^3 - 10x + 15$