

Training session 3 ‘Groups and symmetries in geometry’

Give arguments for all your claims. Write complete sentences, including quantifiers.

- Let n be in \mathbb{N} .
 - Prove that for all $A \in M_n(\mathbb{C})$ the series $e^A := \sum_{m \geq 0} \frac{1}{m!} A^m$ converges in $M_n(\mathbb{C})$.
Hint: choose a norm on $M_n(\mathbb{C})$, for example $\|A\| := \max\{|A_{i,j}| : 1 \leq i, j \leq n\}$, or any other norm (they are all equivalent).
 - Show that for A and B in $M_n(\mathbb{C})$ with $AB = BA$ we have $e^{A+B} = e^A e^B$.
 - Give A and B in $M_n(\mathbb{C})$ such that $e^{A+B} \neq e^A e^B$.
 - Show that $\det(e^A) = e^{\text{tr}(A)}$. Hint: show that it suffices to prove it for diagonalisable A , and then reduce to diagonal A .
- Let G be a Lie group. Show that $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$, $g \mapsto (D\psi_g)(1)$ is a representation of G on \mathfrak{g} . Here $\psi_g: G \rightarrow G$, $h \mapsto ghg^{-1}$, and $(D\psi_g)(1): \mathfrak{g} \rightarrow \mathfrak{g}$ is the derivative of ψ_g at the identity element 1 of G . See also Benthem’s BSc thesis. First consider the case $G = \text{GL}_n$.

This question can be generalised as follows. Let X be a manifold and let G act on X , such that the map $G \times X \rightarrow X$ is a morphism of manifolds. Suppose that G fixes a point x in X , Then G acts on the tangent space $T_X(x)$ and this is a representation of G on $T_X(x)$.
- Let G be a group, let X be a set, and $G \times X \rightarrow X$, $(g, x) \mapsto g \cdot x$ an action of G on X .
 - Let \mathbb{C}^X be the set of all functions $f: X \rightarrow \mathbb{C}$. Show that $G \times \mathbb{C}^X \rightarrow \mathbb{C}^X$, $(g, f) \mapsto g \bullet f$, with, for all $x \in X$, $(g \bullet f)(x) = f(g^{-1}x)$, is an action of G on \mathbb{C}^X .
 - Show that the action in the previous part is linear, where addition and scalar multiplication in \mathbb{C}^X are point-wise ($(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(\lambda \cdot f)(x) = \lambda \cdot (f(x))$), hence makes \mathbb{C}^X into a representation of G .
 - The \mathbb{C} -vector space \mathbb{C}^X is even a \mathbb{C} -algebra: $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$ (point-wise multiplication). Show that the action in the previous part is by \mathbb{C} -algebra automorphisms: for each g in G , the map $\mathbb{C}^X \rightarrow \mathbb{C}^X$, $f \mapsto g \bullet f$ is an isomorphism of \mathbb{C} -algebras.

- (d) Now we take $G = \text{SU}(2)$ and $X = \mathbb{C}^2$, and the action is $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au - \bar{b}v \\ bu + \bar{a}v \end{pmatrix}$. In $\mathbb{C}^X = \mathbb{C}^{\mathbb{C}^2}$ we have the sub- \mathbb{C} -algebra $\mathbb{C}[x, y]$ of polynomial functions $\mathbb{C}^2 \rightarrow \mathbb{C}$, with $x(u, v) = u$ and $y(u, v) = v$. Compute, for $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ in $\text{SU}(2)$, $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \bullet x$ and $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \bullet y$. You should find $\bar{a}x + \bar{b}y$ and $-bx + ay$.
- (e) Conclude that the $\text{SU}(2)$ -action on \mathbb{C}^X preserves $\mathbb{C}[x, y]$, and that

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \bullet (x^j y^k) = (\bar{a}x + \bar{b}y)^j \cdot (-bx + ay)^k.$$

Show that this agrees with the identities on page 10 of Benthem's BSc thesis.

4. On page 12 of Benthem's BSc thesis it is written that $\ker(\text{SU}(2) \rightarrow \text{SO}(3))$ is the intersection of $\text{SU}(2) = \{x \in \mathbb{H} : N(x) = 1\}$ with the center $Z(\mathbb{H}) = \{x \in \mathbb{H} : \forall y \in \mathbb{H}, xy = yx\}$, and that that intersection is $\{1, -1\}$. We provide details.
- (a) Prove that $Z(\mathbb{H}) = \mathbb{R} \cdot 1 \subset \mathbb{H}$.
- (b) Prove that if $x \in \text{SU}(2)$ is such that, for all $y \in \mathbb{R} \cdot I + \mathbb{R} \cdot J + \mathbb{R} \cdot K$, $xyx^{-1} = y$, then $x \in Z(\mathbb{H})$.
5. Prove the formula of Clebsch-Gordan, which means for us, prove that for d_1 and d_2 in $\mathbb{Z}_{\geq 0}$, there is an isomorphism of representations of $\text{SU}(2)$:

$$\mathbb{C}[x, y]_{d_1} \otimes \mathbb{C}[x, y]_{d_2} \longrightarrow \bigoplus_{\substack{d=|d_1-d_2| \\ d \equiv d_1+d_2 \pmod{2}}}^{d_1+d_2} \mathbb{C}[x, y]_d.$$

Hint: prove that both sides have the same character. Try first with small values for d_1 and d_2 . Physicists use this to understand the total angular momentum (around some given direction) of an atom with 2 electrons; see wikipedia.

6. (a) Let g be in $\text{O}(3)$, with $g \neq \text{id}$. Show that the complex eigenvalues λ of g satisfy $|\lambda| = 1$, and that if λ is an eigenvalue, then so is $\bar{\lambda}$. Show that 1 or -1 is an eigenvalue of λ .
- (b) Let g be in $\text{SO}(3)$, with $g \neq \text{id}$. Show that there is a unique $\phi \in [0, \pi]$ such that the complex eigenvalues of g are 1, $e^{i\phi}$ and $e^{-i\phi}$. Deduce from this that there is an oriented orthonormal basis v_1, v_2, v_3 of \mathbb{R}^3 such that g is the rotation about the line $\mathbb{R} \cdot v_3$ over the angle ϕ , and that with respect to the oriented basis $v_2, v_1, -v_3$ g is the rotation about $\mathbb{R} \cdot v_3$ over the angle $-\phi$.
- (c) Make a character table for $\text{SO}(3)$.