Training session 1 'Groups and symmetries in geometry'

1. Recall that $\mathrm{Aff}(\mathbb{R}^2)$ is the group of bijections $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $f(x) = Ax + b$, with $A \in \mathrm{GL}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$.
   (a) Show that the map $F\colon \mathbb{R}^2 \times \mathrm{GL}_2(\mathbb{R}) \to \mathrm{Aff}(\mathbb{R}^2)$ that sends (b, A) to the affine transformation $f_{b,A}\colon x \mapsto Ax + b$ is bijective.
   (b) Compute the group law on $\mathbb{R}^2 \times \mathrm{GL}_2(\mathbb{R})$ obtained, via F, from the group law on $\mathrm{Aff}(\mathbb{R}^2)$.
   (c) The center of a group G is the subset $\{g \in G : \forall_{h\in G}\, gh = hg\}$. Determine the center of $\mathrm{Aff}(\mathbb{R}^2)$.
   (d) Is $\mathrm{Aff}(\mathbb{R}^2)$ isomorphic to the $\mathbb{R}^2 \times \mathrm{GL}_2(\mathbb{R})$ with the product group law $(b_1, A_1)\cdot(b_2, A_2) = (b_1 + b_2, A_1A_2)$?

2. The lines in $\mathbb{R}^2$ are the subsets of the form $\{p+\lambda v : \lambda \in \mathbb{R}\}$, with $p \in \mathbb{R}^2$ and $v \in \mathbb{R}^2 - \{0\}$.
   (a) Draw a picture of a line in $\mathbb{R}^2$ given by a p and v of your choice.
   (b) Let x, y, z in $\mathbb{R}^2$ be distinct and not all 3 on a line. Show that there is a unique $f \in \mathrm{Aff}(\mathbb{R}^2)$ such that $f(x) = 0$, $f(y) = (1, 0)$ and $f(z) = (0, 1)$.

3. The group $\mathrm{Aff}(\mathbb{R})$ is the group of bijections $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f\colon x \mapsto ax + b$. Let $x, y \in \mathbb{R}$ be distinct. Show that there is unique $f \in \mathrm{Aff}(\mathbb{R})$ such that $f(x) = 0$ and $f(y) = 1$.

4. We can do linear algebra with any field F, in particular, for a prime number p, with the field $\mathbb{F}_p$ (also denoted $\mathbb{Z}/p\mathbb{Z}$).
   (a) Define, for any field F, the groups $\mathrm{Aff}(F)$ and $\mathrm{Aff}(F^2)$.
   (b) Let p be a prime number. Show that the groups $\mathrm{Aff}(\mathbb{F}_p)$ and $\mathrm{Aff}(\mathbb{F}_p^2)$ are finite, and find out how many elements they have.

5. Assume that $f \in \mathrm{Sym}(\mathbb{R}^2)$ has the following property: for every line L in $\mathbb{R}^2$, $f(L)$ and $f^{-1}(L)$ are lines. Show that $f \in \mathrm{Aff}(\mathbb{R}^2)$. Actually, this exercise is very hard, so you are not expected to solve it. But at least play a bit with it, and for the next lecture, prepare a question about this. I will ask a few students to state their question.