Exercises sheet 1 for the course Modular Forms
Group Actions in Arithmetic and Geometry
A CIMPA research school on
Gadjah Mada University Yogyakarta, Indonesia

1. Use Cauchy integral formulas to prove that every bounded entire function is constant. (An entire function is a function holomorphic on all of $\mathbb{C}$)

2. Prove that every complex non constant polynomials has a root in $\mathbb{C}$.

3.Let f be meromorphic in Ω, and $a \in \Omega$. prove that $\mathrm{res}_a(f'/f) = \mathrm{ord}_a(f)$ if f is not constantly zero on Ω.

3. Compute the residue of the following function at the specified points

- $f(z) = \frac{z^3-1}{z^2-2z+1}$ at $z = 1$
- $f(z) = \frac{\exp(z)}{z^2-1}$ at $z = 1$
- $f(z) = \frac{\exp(iz)}{z^2+i}$ at $z = i$.
- $f(z) = \frac{1}{z^2+i}^3$ at $z = i$

5. Let f be a meromorphic periodic function. Prove that one of the following holds:

- f is simply periodic, i.e. the periods of f are of the form $n\omega_0$, $n \in \mathbb{Z}$.
- f is doubly periodic i.e. the periods of are of the form $n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbb{Z}$, and ω_1 and ω_2 linearly independent over $\mathbb{R}$.

6. Let Λ be a lattice in $\mathbb{C}$. Prove that if $f \in M(\Lambda)$ then f must have at least one pole.

7. Let Λ be a lattice in $\mathbb{C}$ and Π a fundamental parallelogram for Λ. Prove that if $f, g \in M(\Lambda)$ are such that $\mathrm{ord}_a(f) = \mathrm{ord}_a(g) \quad \text{for all}\, a \in \Pi$, then f/g is constant.

8. Let f be an elliptic function with period lattices Λ. Prove the followings assertions

- $\sum_{a \in \Pi} \mathrm{res}_a(f) = 0$
- $\sum_{a \in \Pi} \mathrm{ord}_a(f) = 0$
- $\sum_{a \in \Pi} \mathrm{ord}_a(f)a \equiv 0 \mod \Lambda$
- An elliptic function cannot have only a simple pole in a fundamental domain.

9. Let $\Lambda \in \mathbb{C}$ be a lattice. Prove that every $f(z)$ can be written in the following form

$$f(z) = R_1(\wp_\Lambda(z)) + \wp'_\Lambda(z) R_2(\wp_\Lambda(z))$$

where R_1 and R_2 rational functions.

10.

(a) Prove that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \overline{\Gamma(1)}$ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \tilde{\mathcal{F}}$ then $c \in \{-1, 0, 1\}$.

(b) Prove that if $\begin{pmatrix} a & b \\ 1 & d \end{pmatrix} = \gamma \in \overline{\Gamma(1)}$ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \tilde{\mathcal{F}}$, then $d \in \{-1, 0, 1\}$.

(c) Prove that if $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \gamma \in \overline{\Gamma(1)}$ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \tilde{\mathcal{F}}$, then $b = -1$ and $a \in \{-1, 0, 1\}$. Moreover, for each $a \in \{-1, 0, 1\}$, described which are the possible τ_1 and τ_2 (note that $\tau_1 = \tau_2$ is admissible).

11. Suppose that τ_1, τ_2 belong to $\mathfrak{h}$ and that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are such that:

$$1 = \alpha(c\tau_1 + d), \quad \tau_2 = \alpha(a\tau_1 + b), \quad 1 = \alpha^{-1}(c'\tau_2 + d'), \quad \tau_1 = \alpha^{-1}(a'\tau_2 + b')$$

for some $\alpha \in \mathbb{C}^*$. Prove that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are inverse of each other.