The goal of these exercises is to compute the character table of the alternating group A_5.

1. Show that all 3-cycles are conjugate in A_5. Show that the 5-cycles are the union of two conjugacy classes. Show that A_5 has 5 conjugacy classes.
2. Show that the only 1-dimensional representation of A_5 is the trivial one.
3. Let ρ_4 denote the representation of A_5 given by permuting the coordinates of the 4-dimensional vector space $\{\mathbf{v} \in \mathbf{C}^5 : \sum_i v_i = 0\}$. Determine the character χ_4 of ρ_4. Show that $\langle \chi_4, \chi_4 \rangle = 1$ and deduce that ρ_4 is irreducible.
4. Show that the only solutions in integers $a, b, c \geq 2$ of the equation

$$1^2 + 4^2 + a^2 + b^2 + c^2 = 60$$

are given by $5, 3, 3$. Deduce that A_5 admits three non-isomorphic irreducible representations ρ_5, ρ_3 and ρ_3' of dimensions 5, 3 and 3 respectively.

This leads to the following partial character table. The second row contains the cardinalities of the corresponding conjugacy classes.

Table 1.

	(1)	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	$(1\,2\,3\,4\,5)^2$
#	1	15	20	12	12
1	1	1	1	1	1
ρ_4	4	0	1	-1	-1
ρ_5	5				
ρ_3	3				
ρ_3'	3				

5. Let G be a finite group and let χ_V be the character of a representation V of G. Show that for any $\sigma \in G$ the numbers $\chi_V(\sigma)$ and $\chi_V(\sigma^{-1})$ are complex conjugate. Show that if $\sigma \in G$ is conjugate to its inverse, then $\chi_V(\sigma) \in \mathbf{R}$.
6. Let G be a finite group and let χ_V be the character of some d-dimensional representation V of G. Show: if $\sigma \in G$ has order m, then $\chi_V(\sigma)$ is a sum of d roots of unity of order dividing m.
7. Show that the missing entries u, v, w in the third column (the one of $(1\,2\,3)$) are in $\mathbf{Z}$. Show that $u^2 + v^2 + w^2 = 1$. Determine the third column. (Hint: use the second orthogonality relation below and show that the 1st and 3rd columns are orthogonal).
8. Show that the missing entries r, s, t in the second column are in $\mathbf{R}$ and satisfy $r^2 + s^2 + t^2 = 3$. Show that they are all equal to ± 1.
9. Let χ_5 denote the characer of ρ_5. Show that $\chi_5(\sigma) = +1$ for $\sigma = (1\,2)(3\,4)$. (Hint: the 2nd and 3rd columns are othogonal).
10. Show that the remaining two entries in the 2nd column are equal to -1. (Hint: the 1st and 2nd columns are orthogonal).
11. Let $x, y \in \mathbf{C}$ be the rightmost two entries in the row of ρ_5. Show that the square of the length of ρ_5 is $\frac{1}{60}(5^2 + 15 + 20 + x\overline{x} + y\overline{y})$. Deduce that $x = y = 0$.

This leads to the following partial character table.

Table 2.

	(1)	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	$(1\,2\,3\,4\,5)^2$
#	1	15	20	12	12
1	1	1	1	1	1
ρ_4	4	0	1	-1	-1
ρ_5	5	1	-1	0	0
ρ_3	3	-1	0		
ρ_3'	3	-1	0		

12. Using orthogonality of various rows and/or columns, show that the 2×2-matrix of the missing four entries is of the form

$$\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix},$$

where $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$. It is a zero of the polynomial $X^2 - X - 1$.

This leads to the following complete character table. The two 3-dimensional representations realize A_5 as a subgroup of $\mathrm{SO}_3(\mathbf{R})$. The A_5-orbit of any non-zero point in $\mathbf{R}^3$ is the set of vertices of a regular icosahedron.

Table 3.

	(1)	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	$(1\,2\,3\,4\,5)^2$
#	1	15	20	12	12
1	1	1	1	1	1
ρ_4	4	0	1	-1	-1
ρ_5	5	1	-1	0	0
ρ_3	3	-1	0	$\frac{1}{2} + \frac{1}{2}\sqrt{5}$	$\frac{1}{2} - \frac{1}{2}\sqrt{5}$
ρ_3'	3	-1	0	$\frac{1}{2} - \frac{1}{2}\sqrt{5}$	$\frac{1}{2} + \frac{1}{2}\sqrt{5}$

Orthogonality relations.

For any two irreducible representations V, V' of a finite group G we have

$$\sum_{\sigma \in G \text{ up to conj.}} \#c(\sigma)\chi_V(\sigma)\overline{\chi_{V'}(\sigma)} = \begin{cases} \#G, & \text{if } V \cong V'; \\ 0, & \text{otherwise.} \end{cases}$$

Here $c(\sigma)$ denotes the conjugacy class of σ.

For all $\sigma, \tau \in G$ we have

$$\sum_{\chi \text{ irr.}} \chi_V(\sigma)\overline{\chi_V(\tau)} = \begin{cases} \#\mathrm{Cent}(\sigma), & \text{if } \sigma \text{ and } \tau \text{ are conjugate;} \\ 0, & \text{otherwise.} \end{cases}$$

Here $\mathrm{Cent}(\sigma)$ denotes the centralizer of σ in G.