Exercise 1. *(1) Determine the center of D_n. (2) Determine the conjugacy classes of D_n. (It might help a little to know that, if n is even, then there are $n/2+3$ conjugacy classes, and if n is odd then there are $(n-1)/2+2$. conjugacy classes.)*

Sol.: Recall that D_n is the group generated by elements r and s, satisfying

$$r^n = s^2 = 1, \qquad sr^k = r^{-k}s, \quad k = 1, \dots, n.$$

(1) If $n = 2$, the goup D_2 is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ and coincides with its center. If n is even an $n > 2$, then the group is non-abelian. In particular no reflection can be in the center. The element $r^{n/2}$ is central: it commutes with the r^k's and also with s, because $r^n = 1$ implies $r^{n/2} = r^{-n/2}$. Since $r^{n/2}$ is the only power of r with this property, it follows that $Z(D_n) = \{1, r^{n/2}\}$.

If n is odd, no power of r is equal to its inverse. Therefore no power of r commutes with s and the center $Z(D_n)$ is trivial.

(2) Let $n > 2$ even.

The identity element and the central element $r^{n/2}$ determine two distinct conjugacy classes. Let $k \neq n/2$. The relations $sr^ks = sr^m\, r^k\, sr^m = r^{-k}$ imply that two elements r^k and r^h are conjugate if and only if $k \equiv \pm h \pmod n$. None of these elements can be conjugate to an element of the form sr^l.

From the relations $sr^m r^k sr^m = sr^{2m-k}$, for $m = 0, \dots, n-1$ and the fact that n is even, it follows that elements sr^k and sr^l are conjugate if and only if $k \equiv l \mod 2$.

In conclusion, when n is even, there are

$$1 + 1 + \frac{n-2}{2} + 2 = \frac{n}{2} + 3$$

conjugacy classes in D_n, namely

$$1, \quad r^{n/2}, \quad \{r^k,\ r^{-k}\},\ k = 1, \dots, \frac{n-2}{2}, \quad \{sr^m,\ m \text{ even}\}, \quad \{sr^m,\ m \text{ odd}\}.$$

Let $n > 2$ odd.

The same arguments as in the even case show that two elements r^k and r^h are conjugate if and only if $h = \pm k \pmod n$. Moreover, since n is odd, any two elements sr^k and sr^l are conjugate. In conclusion, when n is odd, there are

$$1 + \frac{n-1}{2} + 1 = \frac{n+3}{2}$$

conjugacy classes in D_n, namely

$$1, \quad \{r^k,\ r^{-k}\},\ k = 1, \dots, \frac{n-1}{2}, \quad \{sr^m,\ m = 0, \dots, n-1\}.$$

Exercise 2. *If $n \geq 3$, show that the center of S_n is trivial. For $n \geq 4$, show that the center of A_n is trivial.*

Sol.: Let $\sigma \in S_n$ be non-trivial. Then $\sigma(a) = b$ for certain $a \neq b$. Choose c different from a and b. Then $\sigma(bc) \neq (bc)\sigma$. In fact the permutation on the l.h.s. sends a to b, while the one on the r.h.s. sends a to c.

Now let $\sigma \in A_n$ be non-trivial. Then $\sigma(a) = b$ for certain $a \neq b$. Choose c, d such that a, b, c, d are all distinct. Then $\sigma(bcd) \neq (bcd)\sigma$. In fact the permutation on the l.h.s. sends a to b, while the one on the r.h.s. sends a to c.

Exercise 3. *Let $n \geq 1$.*

(a) Let $\sigma \in S_n$ be a product of disjoint cycles c_i of length n_i. Show that σ has order $\mathrm{lcm}(n_i)$.

(b) Exhibit an element of order 6 in S_5.

(c) Exhibit some $n \geq 1$ for which S_n contains an element of order $> n^2$.

Sol.: (a) A cycle of length k has order k. Since disjoint cycles commute, a product of disjoint cycles of length n_i has order $\mathrm{lcm}(n_i)$

(b) The element $(123)(45)$ has order 6 in S_5 (and in S_n, for $n \geq 5$).

(c) For example, we can exhibit integers $p, q, r \geq 1$ with $\gcd(p, q, r) = 1$, so that with $n = p + q + r$ we have $\mathrm{lcm}(p, q, r) = pqr > n^2 = (p + q + r)^2$.

Exercise 4. *Let $n \geq 1$ and let $\sigma \in A_n$. Let C denote the conjugacy class of σ in $\mathcal{S}_n$. So we have $C = \{\tau\sigma\tau^{-1} : \tau \in \mathcal{S}_n\}$.*

(a) Show that $C \subset A_n$.

(b) Show that either C is a conjugacy class of A_n or it is a union of two conjugacy classes.

(c) Show that C is a conjugacy class of A_n if and only if there is an odd permutation in the $\mathcal{S}_n$-centralizer of σ.

Sol.: (a) We have that $sign(\tau\sigma\tau^{-1}) = sign(\tau)sign(\sigma)sign(\tau) = sign(\sigma)$. Hence $C \subset A_n$.

(b) One has $S_n/A_n \cong \mathbf{Z}_2$. So $S_n = A_n \cup A_n\tau_0$, with $\tau_0 \in S_n \setminus A_n$, and the conjugacy class of σ in S_n is given by

$$C = \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau\sigma\tau^{-1} : \tau \in A_n\} \cup \{\tau\tau_0\sigma\tau_0^{-1}\tau^{-1} : \tau \in A_n\}.$$

(c) C is a conjugacy class of A_n if and only if $\tau_0\sigma\tau_0^{-1}$ is conjugate to σ in A_n if and only if there exists $\gamma \in A_n$ such that

$$\tau_0\sigma\tau_0^{-1} = \gamma\sigma\gamma^{-1} \quad \Leftrightarrow \quad \tau_0 = \gamma\zeta, \text{ for some } \zeta \in \mathbf{Z}_{S_n}(\sigma),$$

$$\Leftrightarrow \quad \gamma = \tau_0\zeta^{-1}.$$

Since τ_0 is odd, so has to be ζ.

Exercise 5. *Let $n \geq 1$. Show that $\mathbf{Z}/n\mathbf{Z}$ admits n distinct 1-dimensional representations.*

Sol.: Let C denote a cyclic group of order n and let $w \in C$ be a generator. Let $\zeta \in \mathbf{C}^*$ be a primitive n^{th} root of 1. For $k \in \mathbf{Z}$ the maps $\phi_k : C \to \mathbf{C}^*$, determined by $\phi_k(w) = \zeta^k$, are homomorphisms and define 1-dimensional representations of C_n. These representations are mutually non-isomorphic because the corresponding characters χ_k are distinct: indeed $\chi_k = \phi_k$, for all $k \in \mathbf{Z}$ and $\phi_k = \phi_{k'}$ if and only if $k \equiv k' \pmod{n}$.

Exercise 6. *Let $H = \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ in A_4.*
(a) Write down the multiplication table of the elements in H and deduce that H is a normal subgroup of A_4 that is isomorphic to Klein's 4-group.
(b) Show that H is the commutator subgroup of A_4.
(c) Show that A_4 admits three distinct 1-dimensional representations.

Sol.: All non-trivial elements of H have order two. Moreover

$$\begin{aligned}
(1\,2)(3\,4)\cdot(1\,3)(2\,4) &= (1\,4)(2\,3), \quad & (1\,2)(3\,4)\cdot(1\,4)(2\,3) &= (1\,3)(2\,4),\\
(1\,3)(2\,4)\cdot(1\,4)(2\,3) &= (1\,2)(3\,4), \quad & (1\,3)(2\,4)\cdot(1\,2)(3\,4) &= (1\,4)(2\,3),\\
(1\,4)(2\,3)\cdot(1\,2)(3\,4) &= (1\,3)(2\,4), \quad & (1\,4)(2\,3)\cdot(1\,3)(2\,4) &= (1\,2)(3\,4).
\end{aligned}$$

Hence H is a group isomorphic to Klein's 4-group V_4. It is clearly normal in A_4, because conjugation preserves the cycle type.
(b) The quotient group A_4/H is a group with 3 elements: hence it is abelian and isomorphic to $\mathbf{Z}_3$. This implies the inclusion $[A_4, A_4] \subset H$. On the other hand, $[A_4, A_4] \neq 1$, because A_4 is not abelian. In addition, $[A_4, A_4]$ cannot one of the order 2 subgroups of H, because A_4 has trivial center and hence no normal subgroups of order 2 (a normal subgroup of order 2 is necessarily central). In conclusion, $[A_4, A_4] = H$.

Exercise 7. *Show that every finite group G has a faithful representation, i.e., there is a representation (π, V) such that the homomorphism $\pi : G \to GL(V)$ is injective. (One can rephrase this as "every finite group is a subgroup of $GL(n, \mathbf{C})$ for some n".)*

Sol.: Consider the regular representation $G \to GL(\mathbf{C}[G])$

$$g \cdot (\sum_{\gamma \in G} z_\gamma \gamma) = \sum_{\gamma \in G} z_\gamma g\gamma.$$

This representaion is faithful, i.e. there is no element $g \in G$ which fixes all elements in $\mathbf{C}[G]$. Indeed, if that were the case, then there would exists $\gamma \in G$ such that $g\gamma = \gamma$. This can be true if and only if $g = id$.

Exercise 8. *Let $A : \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ be the map given by $A(x, y) = (y, x)$.*
(a) Show that $A^2 = \mathrm{id}$ and let G denote the group $\{I, A\}$.
(b) Write this representation $r : G \longrightarrow \mathrm{GL}_2(\mathbf{C})$ as a product of two 1-dimensional representations.

Sol.: (a) One has $A(A(x, y)) = A(y, x) = (x, y)$, for all $(x, y) \in \mathbf{C}^2$.
(b) Every vector $Z \in \mathbf{C}^2$ can be written as $Z = \frac{1}{2}(Z + AZ) + \frac{1}{2}(Z - AZ)$. So $\mathbf{C}^2 = V_1 \oplus V_{-1}$, where $V_{\pm 1}$ are the ± 1 eigenspaces of A in $\mathbf{C}^2$. Both are 1-dimensional.