In this note we determine the automorphism groups of the symmetric groups S_n. For $n = 2$ this is very easy: we have $S_2 \cong \mathbf{Z}_2$ and hence $\mathrm{Aut}(S_2)$ is trivial. Therefore we suppose from now on that $n > 2$. The main result is Theorem 8.

For the convenience of the reader we first recall some basic properties of the groups S_n and the subgroups A_n of *even* permutations.

Lemma 1. *Let $n > 2$.*
(a) The center of S_n is trivial;
(b) For $n > 3$ the center of A_n is trivial.

Proof. (a) Let $\sigma \in Z(S_n)$. If $\sigma \neq \mathrm{id}$, then there exist two distinct $a, b \in \{1, 2, \ldots, n\}$ with $\sigma(a) = b$. Choose $c \in \{1, 2, \ldots, n\}$ with $c \neq a$ and $c \neq b$. Then $(b\,c)\sigma \neq \sigma(b\,c)$ because $(b\,c)\sigma$ maps a to c, while $\sigma(b\,c)$ maps a to b. This shows that $\sigma = \mathrm{id}$ and $Z(S_n)$ must be trivial, as required.
(b) Similarly, suppose that $\sigma \in Z(A_n)$ is non-trivial. Pick two distinct $a, b \in \{1, 2, \ldots, n\}$ with $\sigma(a) = b$ and choose two elements $c, d \in \{1, 2, \ldots, n\}$ different from a and b. Then $(b\,c\,d)\sigma \neq \sigma(b\,c\,d)$ because the two permutations map a to different elements.

Lemma 2. *Two elements of S_n are conjugate if and only if they have the same cycle type.*

Proof. For any $\sigma \in S_n$ and any $d \leq n$ we have

$$\sigma(1\,2\,\ldots\,d)\sigma^{-1} \;=\; (\sigma(1)\,\sigma(2)\,\ldots\,\sigma(d)).$$

This shows that any conjugate of a d-cycle is again a d-cycle. Since every permutation is a product of disjoint cycles, it follows that the cycle types of conjugate permutations are the same. In the other direction, let $\tau = (a_1 \ldots a_r)(a_{r+1} \ldots a_s) \ldots (a_l \ldots a_m)$ and $\tau' = (a'_1 \ldots a'_r)(a'_{r+1} \ldots a'_s) \ldots (a'_l \ldots a'_m)$ be two permutations having the same cycle type. Define $\sigma \in S_n$ by $\sigma(a_i) = a'_i$ for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned}
\sigma\tau\sigma^{-1} &= \sigma(a_1 \ldots a_r)\sigma^{-1}\sigma(a_{r+1} \ldots a_s)\sigma^{-1} \ldots \sigma(a_l \ldots a_m)\sigma^{-1},\\
&= (a'_1 \ldots a'_r)(a'_{r+1} \ldots a'_s) \ldots (a'_l \ldots a'_m),\\
&= \tau'.
\end{aligned}$$

This shows that τ and τ' are conjugate, as required.

Lemma 3. *Let $n > 2$.*
(a) The group A_n is generated by 3-cycles.
(b) Any normal subgroup of A_n that contains a 3-cycle, is equal to A_n itself.

Proof. (a) The product $(1\,2)(2\,3)$ is equal to the 3-cycle $(1\,2\,3)$. The product of two disjoint 2-cycles $(a\,b)$ and $(c\,d)$ is equal to $(a\,b)(b\,c)(b\,c)(c\,d)$ and is hence a product of two 3-cycles. Since any element of A_n is a product of an *even* number of transpositions, it is therefore a product of 3-cycles.
(b) Let $N \subset A_n$ be a normal subgroup and suppose that $(1\,2\,3) \in N$. Let $\sigma' \in A_n$ be an arbitrary 3-cycle. Then $\sigma' = \tau(1\,2\,3)\tau^{-1}$ for some $\tau \in S_n$. If $\tau \in A_n$, then $\sigma' \in N$ and we are done. If not, then $\tau' = \tau(1\,2)$ is in A_n and $\sigma' = \tau'(1\,3\,2)\tau'^{-1}$ is once again in N.

Lemma 4. *The commutator subgroup of S_n is equal to A_n. For $n \geq 5$ the commutator subgroup of A_n is equal to A_n itself.*

Proof. Since S_n/A_n is commutative, the commutator subgroup S_n' is contained in A_n. Conversely, we have $(1\,2)(1\,3)(1\,2)^{-1}(1\,3)^{-1} = (1\,2\,3)$, showing that every 3-cycle is in S_n'. By Lemma 3 (a) the group A_n is generated by 3-cycles, so that $S_n' = A_n$ as required.

The identity
$$(1\,2\,3)(3\,4\,5)(1\,2\,3)^{-1}(3\,4\,5)^{-1} \;=\; (1\,4\,3).$$
shows that for $n \geq 5$ every 3-cycle is a commutator of A_n. This implies the second statement.

We remark that the group A_3 is abelian, so that its commutator subgroup is trivial. The group A_4 is not abelian. Its commutator subgroup is
$$V_4 \;=\; \{(1),\,(1\,2)(3\,4),\,(1\,3)(2\,4),\,(1\,4)(2\,3)\}.$$
Indeed, V_4 is normal and the quotient A_4/V_4 has order 3 and is hence abelian. It follows that $A_4' \subset V_4$. Equality follows from the identity $(1\,2\,3)(1\,2\,4)(1\,2\,3)^{-1}(1\,2\,4)^{-1} = (1\,2)(3\,4)$.

Proposition 5. *Let $n \geq 5$. Then the group A_n is simple, i.e. does not contain any proper normal subgroups. The only proper normal subgroup of S_n is A_n.*

Proof. Let $N \subset A_n$ be a non-trivial normal subgroup. We will show that N contains a 3-cycle. Then Lemma 3 (b) implies the required result.

Step 1. Suppose that N contains a permutation σ which is a product of disjoint cycles at least one of which has length $d \geq 4$. Then, up to renumbering, we have $\sigma = (1\,2\,\ldots\,d)\tau$ where τ leaves $\{1, 2, \ldots, d\}$ invariant. The permutation $\sigma^{-1}(1\,2\,3)\sigma(1\,2\,3)^{-1}$ is contained in N. One easily checks that it is equal to the 3-cycle $(1\,3\,d)$.

Step 2. This leaves us with the possibility that all permutations in N are products of disjoint cycles of length ≤ 3. Suppose that N contains a permutation σ admitting a 3-cycle. If it admits *only one* 3-cycle, then its square is a 3-cycle and we are done. If it contains *at least two* 3-cycles, we may assume that $\sigma = (1\,2\,3)(4\,5\,6)\tau$ where τ leaves $\{1, 2, \ldots, 6\}$ invariant. Then $\sigma^{-1}(1\,2\,4)\sigma(1\,2\,4)^{-1}$ is contained in N. One easily checks that it is equal to $(1\,4\,2\,3\,6)$ and we are done by Step 1.

Step 3. This leaves us with the possibility that all permutations in N are products of disjoint transpositions. Let $\sigma \in N$ be a non-trivial element. Since σ is even, it is a product of at least two transpositions and we may assume that $\sigma = (1\,2)(3\,4)\tau$, where τ leaves $\{1, 2, 3, 4\}$ invariant. Then $\sigma(1\,2\,3)\sigma(1\,2\,3)^{-1} = (1\,3)(2\,4)$ is in N. Since $n \geq 5$ the permutation $(1\,3)(2\,4)(1\,3\,5)(1\,3)(2\,4)(1\,3\,5)^{-1}$ is in N. It is equal to the 3-cycle $(1\,3\,5)$ and we are done.

To prove the second statement of the Proposition, let N be a proper normal subgroup of S_n. Then $N \cap A_n$ is a normal subgroup of A_n. So either $N \subset A_n$ in which case $N = \{1\}$ or $N = A_n$ or we have $N \cap A_n = \{1\}$. In the latter case $\#N \leq 2$ and hence $N \subset Z(S_n)$. Lemma 1 implies then that $N = \{1\}$. This proves the proposition.

We remark that the possibility that arises in Step 3 of the proof of Lemma 5, actually occurs for $n = 4$. In that case the group V_4 mentioned above is a normal subgroup of A_4. Its elements are products of disjoint transpositions.