Exercise sheet nb. 2 February 28, 2020

In all this sheet of exercises, we denote by $M_k = M_k(SL_2(\mathbb{Z}))$ the set of modular forms of weight k for the full modular group and by S_k the subset of cusp forms in M_k.

Exercice 1. Use the *formula* $k/12$ to show that E_4 has a simple zero at ρ and no other zero in $\mathcal{H}$ and E_6 has a simple zero at i and no other zero in $\mathcal{H}$.

Exercice 2.

1. Let $f \in M_k$ and $g \in M_\ell$. Show that fg defines a modular form of weight $k+\ell$. Deduce that $M = \oplus_k M_k$ is a graded algebra.
2. Show that the map $X \mapsto E_4$; $Y \mapsto E_6$ defines an isomorphism between $\mathbb{C}[X,Y]$ and M. (So we will make the identification with the polynomial algebra in $E_4, E_6 : M = \mathbb{C}[E_4, E_6]$.)

Exercice 3. Show that $M_{14} = \mathbb{C}E_{14}, S_{14} = \{0\}$ and $E_{14} = E_6E_8 = E_6E_4^2$.

Exercice 4 (Bernoulli numbers)**.** Recall that the Bernoulli numbers B_k are defined by the power series:

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} \frac{B_k}{k!} x^k.$$

1. Compute B_0, B_1, B_2. Show that $B_k = 0$ if $k > 1$ is odd.
2. Using Cauchy product formula, show the induction relation: $(n+1)B_n = -\sum_{k=0}^{n-1} \binom{n+1}{k} B_k$.

Exercice 5 (an Euler's formula)**.** The goal of this exercise is to show Euler's formula:

$$\zeta(k) = -\frac{(2i\pi)^k}{2k!} B_k \quad (k \geq 4, even) \tag{1}$$

where B_k is the k-th Bernoulli number.

1. We will first show the Euler's lemma: for all $z \in \mathbb{C}\backslash\mathbb{Z}, \pi\cot(\pi z) = \frac{1}{z} + \sum_{m\geq 1} \left(\frac{1}{z+m} + \frac{1}{z-m}\right)$. Let's denote by $f(z) = \pi\cot(\pi z)$ and by $g(z) = \frac{1}{z} + \sum_{m\geq 1} \left(\frac{1}{z+m} + \frac{1}{z-m}\right)$.
   (a) Show that f and g are holomorphic over $\mathbb{C}\backslash\mathbb{Z}$, and have simple poles of residue 1 at every integer.
   (b) Prove that f and g are odd and 1-periodic.
   (c) Deduce that $h = f - g$ is holomorphic over $\mathbb{C}$, odd and 1-periodic.
   (d) Show that $h(z)$ is bounded for $\Im(z) \to \infty$.
   (e) Deduce that h is bounded on $\mathbb{C}$ and conclude.
2. Use Euler's lemma to show that $y\cot(y) = 1 - 2\sum_{k\geq 1} \frac{\zeta(2k)}{\pi^{2k}} y^{2k}$.
3. Prove that for $\tau = 2iy$, we also get: $y\cot(y) = \frac{\tau}{2} + \sum_{k\geq 1} \frac{B_k}{k!}\tau^k$.
4. Conclude to Euler's formula.

Exercice 6. Let $n \in \mathbb{Z}, n > 0$. Recall the q-expansion: $E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1} \sigma_{k-1}(n) q^n$.

1. Verify that the q-expansion of E_k for $k \leq 14$ is the following:

$$E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \qquad E_{10}(z) = 1 - 264 \sum_{n \geq 1} \sigma_9(n) q^n$$

$$E_6(z) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n \qquad E_{12}(z) = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n$$

$$E_8(z) = 1 + 480 \sum_{n \geq 1} \sigma_7(n) q^n \qquad E_{14}(z) = 1 - 24 \sum_{n \geq 1} \sigma_{13}(n) q^n.$$

2. Prove that $\sigma_7(n) \equiv \sigma_3(n) \pmod{120}$.
3. Prove that $11\sigma_9(n) = -10\sigma_3(n) + 21\sigma_5(n) + 5040 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_5(n-j)$.
4. Prove that

$$E_6^2 - E_{12} = -\frac{762048}{691}\Delta.$$

Using the factorisations $504 = 2^3.3^2.7, 65620 = 2^4.3^2.5.7.13, 762048 = 2^6.3^5.7^2$, deduce that

$$756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 252.691 \sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j) \quad (n \geq 1).$$

Deduce the Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$