Excercises 1

1. Let L be a lattice in $\mathbb{R}^n$ with the usual inner product.
(a) Check that $\mathrm{covol}(L)$ does not depend on the choice of the basis used to compute it.
(b) Compute the covolume of the lattice $L = \mathrm{span}_{\mathbb{Z}}\{\mathbf{w}_1, \mathbf{w}_2\}$ in $\mathbb{R}^2$, where
$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} -100 \\ 1 \end{pmatrix}.$$
Show that $L = \mathbb{Z}^2$.
(c) Check that $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} \cos\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} \end{pmatrix}$ is a basis of the hexagonal lattice in $\mathbb{R}^2$. Compute its covolume.

2. Let
$$L = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3 \ : \ x + y + z \equiv 0 \mod 7\}.$$
Show that $L \subset \mathbb{R}^3$ is a lattice and compute its covolume.

3. Let L be a lattice. Let A be an invertible $n \times n$ matrix.
(a) Show that $A(L)$ is a lattice.
(b) Show that $covol(A(L)) = |\det(A)| covol(L)$.
(c) Let $m \in \mathbb{R}_{>0}$. Show that $covol(mL) = m^n covol(L)$.

4. Consider
$$L = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 : \text{either } x, y, z, w \in \mathbb{Z} \text{ or } x, y, z, w \in \tfrac{1}{2} + \mathbb{Z}\}.$$
(a) Show that L is a lattice in $\mathbb{R}^4$, exhibit a $\mathbb{Z}$-basis of L, and compute the covolume of L.
(b) Show that the sum of the coordinates of any element of L is an integer. Show that the elements whose sum of the coordinates is even is a sublattice L' of L. Compute the covolume of L'.

5. Check that the ball $B_2^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 < 2\}$ is convex and symmetric with respect to the origin. How many non-zero vectors of the lattice $\mathbb{Z}^n$ does it contain?

6. Let L be a lattice of full rank in $\mathbb{R}^n$, and let $M \subset \mathbb{R}^n$ be a set such that $\mathrm{vol}(M) < \mathrm{covol}(L)$. Prove that there exists a point $\mathbf{x} \in \mathbb{R}^n$ such that the intersection $M \cap (L + \mathbf{x})$ is empty.

7. Let L be the integer lattice $\mathbb{Z}^2$. Let $Q_\epsilon = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| < 1,\ |y| < 1+\epsilon\}$, with $\epsilon \geqslant 0$.
(a) Show that for $\epsilon = 0$, the square Q_0 contains no non-zero lattice points;
(b) Show that there exists $\epsilon > 0$ such that Q_ϵ contains precisely 2 non-zero lattice points.

8. Let L be the integer lattice $\mathbb{Z}^2$. Let $B_\epsilon = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 + y^2 < (1+\epsilon)^2\}$.
(a) Show that for $\epsilon = 0$, the disk B_0 contains no non-zero lattice points;
(b) Show that there exists $\epsilon > 0$ such that Q_ϵ contains precisely 4 non-zero lattice points.