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Hypergeometric motives: Hodge numbers and supercongruences

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Consider the Dwork family of quintic threefolds in $\mathbb{P}^4$

$$X_\psi : x_1 + x_2 + x_3 + x_4 + x_5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0 \tag{1}$$

where $\psi \in \mathbb{C}$ is a parameter. The non singular member of the family are Calabi-Yau varieties. Recall that a complex connected, compact Kähler manifold X is called a Calabi-Yau variety if

(CY1) the canonical bundle is trivial

(CY2) there are no p-holomorphic forms for $p \neq 0, n$, where n is the complex dimension of X.

Smooth hypersurfaces of degree $n + 1$ in a n-dimensional projective space are Calabi-Yau as consequence of adjunction and the Lefschetz theorem. Elliptic curve and $K3$ surfaces are the only examples of Calabi-Yau variety in dimension 1 and 2. It is not difficult to show that if ψ is not a fifth root unity then X_ψ is smooth and hence is a Calabi-Yau threefold. Let

$$P_\psi(x_1, \ldots, x_5) = x_1 + x_2 + x_3 + x_4 + x_5 - 5\psi x_1 x_2 x_3 x_4 x_5$$

The Hodge diamond of X_ψ looks a follows

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

In particular $H^1(X_\psi, \mathbb{C}) = H^5(X_\psi, \mathbb{C}) = 0$ and $H^0(X_\psi, \mathbb{C})$, $H^2(X_\psi, \mathbb{C})$, and $H^4(X_\psi, \mathbb{C})$ are one dimensional.

Suppose that $\psi \in \mathbb{Q}$ then we can reduce modulo a prime p primes not dividing its denominator; assume $p > 5$.For varieties over finite fields an important quantity is the number of points

$$N_p(X_\psi) = \# \left\{ P \in \mathbb{P}^4(\mathbb{F}_p) \mid P \in X_\psi \right\}.$$

Then if we set

$$A_p = \sum_{(x_1, \ldots, x_5) \in \mathbb{F}_p^5} \left(1 - P_\psi(x_1, \ldots, x_5)^{p-1} \right) \tag{2}$$

we have that

$$A_p \equiv 1 - (p-1) N_p(X_\psi) \bmod p$$

hence

$$A_p - 1 \equiv N_p(X_\psi) \bmod p.$$

A prime is called a good prime for X_ψ if the reduction of X_ψ mod p is non singular. Next we want to introduce the generalized hypergeometric functions. Let r and s be integers, and $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ be rational number with all the $-\beta_i$ different from non-negative integers. The generalized hypergeometric function is defined as the series

$${}_rF_s\left(\begin{matrix} \alpha_1 & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s \end{matrix} \Big| t \right) == \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_r)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{t^k}{k!}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$. We want to compute $A_p \bmod p$ in terms of truncated generalized hypergeometric function. First of all recall that

$$\sum_{x\in\mathbb{F}_p^*} x^a = \begin{cases} p-1 & \text{if } a \equiv 0 \bmod p \\ 0 & \text{if } a \not\equiv 0 \bmod p \end{cases}$$

Expanding the polynomial in (2) and using the above relation it is not hard to show that:

$$A_p \equiv \sum_{m=0}^{\lfloor p/5 \rfloor} \frac{(5m)!}{m!^5}\left(\frac{t}{5^5}\right)^m \bmod p$$

where $t = \psi^{-5}$. See [1, p. 38] for more details. On the right hand side we have the truncatation of the generalized hypergeometric function:

$$\sum_{m=0}^{\infty} \frac{(5m)!}{m!^5}\left(\frac{t}{5^5}\right)^m = {}_4F_3\left(\begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 1 & 1 & 1 & \end{matrix} \Big| t\right)$$

Another important aspect of this theory are the periods of X_ψ. Recall that, roughly speaking, a period on a k-dimensional variety (defined over $\overline{\mathbb{Q}}$) is the value of the integral along a k-cycle (with some boundary condition) of a k-differential form (for more about periods and their relevance in arithmetic geometry see [2]). Note that $(CY1)$ implies that X_ψ admits a nowhere-vanishing holomorphic 3-form ω_ψ, unique up to scalar multiplication, which can be defined as:

$$\mathrm{Res}_{X_\psi}\left(\frac{\sum_{i=1}^5 (-1)^i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_5}{x_1 + x_2 + x_3 + x_4 + x_5 - 5\psi x_1 x_2 x_3 x_4 x_5}\right)$$

Therefore the periods of X_ψ, are the values of $\int_\gamma \omega_\psi$ where γ is a 3-cycle. Moreover the dimension of $H^3(X_\psi, \mathbb{C}) = 204$ which means there are 204 periods of ω_ψ.

Consider the abelian subgroup of automorphisms

$$A := \{(\zeta_1, \ldots, \zeta_5) | \zeta_i^5 = 1, \zeta_1 \cdots \zeta_5 = 1\},$$

acting by $x_i \mapsto \zeta_i x_i$ and let V_ψ be the subspace of $H^3(X_\psi, \mathbb{C})$ fixed by A. It can be shown that the dimension of V_ψ is 4.

There is also another way to compute the number of points of X_ψ mod p which is done via the Frobenius endomorphism and goes as follows: For a good prime p, let Frob_p denote the Frobenius morphism on X_ψ induced by the p-th power map $x \to x^p$. Let ℓ be a prime different from p. Then the induced operator Frob_p^* acts on the ℓ adic étale cohomology groups $H^i_{et}(X_\psi, \mathbb{Q}_\ell)$. Let

$$P_{p,i}(T) := \det(1 - T\,\mathrm{Frob}_p^* \,| H^i_{et}(X_\psi, \mathbb{Q}_\ell)$$

be the characteristic polynomial of the endomorphism Frob_p^* on the étale ℓ-adic cohomology group, where T is an indeterminate. By the Lefschetz fixed point formula we have that

$$N_p(X_\psi) = \#\,\mathrm{Fix}(\mathrm{Frob}_p) = \sum_i (-1)^i \,\mathrm{Tr}(\mathrm{Frob}_p^* \,| H^i_{et}(X_\psi, \mathbb{Q}_\ell)).$$

Recall that we set $t = \psi^{-5}$. If $t \to 1$ (i.e. $\psi \to$ a root of unitiy), then dimension of V_ψ^I the fixed part of V_ψ under the action of the inertia group has dimension 3 and split as the direct sum of L and A, where L has dimension 1 and A has dimension 2 and is associated to a modular form f of weight 4 and level 125 (as proven by C. Schoen in [6]). Moreover

$$\text{Trace of Frobenius on } V = \left(\frac{p}{5}\right) p + a_p,$$

where a_p is the $p-th$ coefficent of the modular form found by Schoen and $\left(\frac{p}{5}\right)$ is the Legendre symbol. It follows that

$$\sum_{m=0}^{\lfloor p/5 \rfloor} \frac{(5m)!}{m!^5} 5^{-5m} \equiv a_p \bmod p$$

Experimentally this congruence actually happens mod p^3 (but only for $t = 1$), which is rather surprising, and this fact has been given the name of *supercongruence*. One can find a few other examples of this kind in [5] (for a recent proof see [3]).

Together with D. Roberts we have found a conjectural explanation of this supercongruence phenomenon tying it to the gap in the Hodge numbers of the limiting motive at $t = 1$. In our case this motive is that of the modular form f, which being of weight 4 has Hodge numbers $(1, 0, 0, 1)$. The gap of 3 between the non-zero Hodge numbers should explain the observed supercongruence to the power p^3. For more details see [4].

References

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