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A Rigidity theorem for translates of uniformly convergent Dirichlet series

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In 1975, Voronin [6] discovered the following universality property of the Riemann zeta function $\zeta(s)$: given an holomorphic and non-vanishing function $f(s)$ on a closed disk K inside the critical strip $\frac{1}{2} < \sigma < 1$, for every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \frac{1}{2T}|\{\tau \in [-T,T] : \max_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\}| > 0. \qquad (1)$$

Voronin's universality theorem has been extended in several directions, in particular involving other L-functions in place of $\zeta(s)$ or vector of L-functions in place of a single L-function and other compact sets in place of disks; see the survey by Matsumoto [3] and Chapter VII of Karatsuba-Voronin [2]. Those results cannot yet be valid in the region $\sigma > 1$, since every Dirichlet series $F(s)$ is Bohr almost periodic and bounded on any vertical strip whose closure lies inside the half plane of uniform convergence $\sigma > \sigma_u(F)$.

We recall that a general Dirichlet series (D-series for short) is of the form

$$F(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda_n s} \tag{2}$$

with coefficients $a(n) \in \mathbb{C}$ and a strictly increasing sequence of real exponents $\Lambda = (\lambda_n)$ satisfying $\lambda_n \to \infty$. The case $\lambda_n = \log n$ recover the ordinary Dirichlet series. A basis for a D-series is a sequence of real numbers $B = (\beta_l)$ that satisfies the following three conditions:
-the elements of B are $\mathbb{Q}$-linearly independent;
-every λ_n is a $\mathbb{Q}$-linear combination of elements of B;
- every β_l is a $\mathbb{Q}$-linear combination of elements of Λ.
This can be expressed in matrix notation by considering Λ, B as column vectors, and writing the last two conditions as $B = T\Lambda, \Lambda = RB$, where R, T are some Bohr matrices, whose row entries are rational numbers and almost always 0; R is uniquely determined by Λ and B. We say that two D-series $F(s) = \sum_{n\geq 1} a(n)e^{-\lambda_n s}, G(s) = \sum_{n\geq 1} b(n)e^{-\lambda_n s}$, with same exponents Λ, are equivalent if there exist a basis B of Λ and a real column vector $Y = (y_l)$ such that

$$b(n) = a(n)e^{i(RY)_n}, \tag{3}$$

where R is the Bohr matrix related to Λ and B. We observe that in the case of ordinary Dirichlet series with coefficients $a(n), b(n)$, the equivalence relation reduces to the existence of a completely multiplicative function $\rho(n)$ such that $b(n) = a(n)\rho(n)$ for all $n \geq 1$ and such that $|\rho(p)| = 1$ if $p|n$ and $a(n) \neq 0$.
We say that a D-series $F(s)$ or a sequence of exponents Λ has an integral basis if there exists a basis B of Λ such that the associated Bohr matrix R has integer entries. Such basis B is called an integral basis of $F(s)$ or of Λ. Note that an ordinary Dirichlet series ($\Lambda = (\log n)$) has an integral basis ($B = (\log p)$).
We extend the notion of equivalence to vectors in the following way: let $N \geq 1$ and for $j = 1, \ldots, N$ let $F_j(s), G_j(s)$ be two D-series with coefficients $a_j(n), b_j(n)$ respectively and the same exponents Λ. We say

that the vectors $(F_1(s), ..., F_N(s))$ and $(G_1(s), ..., G_N(s))$ are equivalent if exist a basis B of Λ and a real vector $Y = (y_l)$ such that

$$b_j(n) = a_j(n)e^{i(RY)_n}, \forall j = 1, ..., N \tag{4}$$

where R is the Bohr matrix related to Λ and B.
Before state the main result we recall a fundamental result of Bohr theory; see Bohr [1].

Theorem 1 (Bohr's equivalence theorem) *Let $F(s), G(s)$ be equivalent D-series with abscissa of absolute convergence σ_a. Then in any open half plane $\sigma > \sigma_1 > \sigma_a$ the functions $F(s), G(s)$ take the same set of values.*

Now we state the main result; see the paper of Perelli-Righetti [4].

Theorem 2 (Perelli-Righetti) *Let $N \geq 1$ and, for $j = 1, ..., N$, let $F_j(s)$ be general D-series with coefficients $a_j(n)$ and the same exponents Λ, with an integral basis and with finite abscissa of uniform convergence $\sigma_u(F_j)$. Further, let K_j be compact sets inside the half planes $\sigma > \sigma_u(F_j)$ containing at least one accumulation point and let $f_j(s)$ be holomorphic on K_j. Then the following assertions are equivalent:*
i) For every $\varepsilon > 0$ there exists $\tau \in \mathbb{R}$ such that

$$\max_{j=1,...,N} \max_{s \in K_j} |F_j(s + i\tau) - f_j(s)| < \varepsilon; \tag{5}$$

ii) $f_1(s), ..., f_N(s)$ are D-series with exponents Λ, and $(f_1(s), ..., f_N(s))$ is vector equivalent to $(F_1(s), ..., F_N(s))$;
iii) for every $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{2T} |\{\tau \in [-T, T] : \max_{j=1,...,N} \max_{s \in K_j} |F_j(s + i\tau) - f_j(s)| < \varepsilon\}| > 0; \tag{6}$$

iv) $f_j(s)$ has analytic continuation to $\sigma > \sigma_u(F_j)$ and there exists a sequence τ_k such that $F_j(s + i\tau_k)$ converges uniformly to $f_j(s)$ on every closed vertical strip in $\sigma > \sigma_u(F_j)$, $j = 1, ..., N$.

Note that Theorem 2 holds for ordinary Dirichlet series. Moreover, note that it represents the counterpart of the universality theorems of L-functions in the critical strip. Indeed, Theorem 2 gives a complete characterization of the analytic functions $f_j(s)$ approximable by such translates as in i) and, by Theorem 1 and its converse for D-series with an integral basis (see Righetti [5]), we see that such functions $f_j(s)$ are those assuming the same set of values of the $F_j(s)$'s on any vertical strip inside the domain of absolute convergence. Finally, thanks to iv), such $f_j(s)$'s have analytic continuation to $\sigma > \sigma_u(F_j)$.
We conclude with some remarks about the relevance of integral bases in Theorem 2. Arguing in a similar way as in the proof of Theorem 1 and Theorem 2 one can prove the following

Theorem 3 *Under the assumption of Theorem 2, with Λ not necessarily having an integral basis, suppose that i) holds. Then the $f_j(s)$'s are D-series with coefficients $b_j(n)$ and the same exponents Λ satisfying the following properties: for $j = 1, ..., N$*

$$|b_j(n)| = |a_j(n)|, \sigma_u(f_j) = \sigma_u(F_j) \tag{7}$$

and the set of values of f_j and of F_j on any open vertical strip inside $\sigma > \sigma_u(F_j)$ coincide. Moreover, i) holds for the $f_j(s)$'s described in ii) of Theorem 2.

Similar remarks and variants, namely without assuming the existence of an integral basis, apply also to the equivalence of i) with iii) and iv) in Theorem 2. However, $f_j(s)$ may not be equivalent to $F_j(s)$, as shown by the following example by Bohr [2, pp.151-153]. Let

$$\lambda_n = 2n - 1 + \frac{1}{2(2n-1)}, F(s) = \sum_{n=1}^{\infty} e^{-\lambda_n s}, f(s) = -F(s). \tag{8}$$

In this case, since every λ_n is rational, all bases B of Λ consist of a single rational number, and since the least common multiple of the denominators of the λ_n is ∞, no one is an integral basis. Moreover, the Bohr matrix R, relative to Λ and B reduces to an infinite column vector, hence the vectors Y reduce to a single real number; thus the set of D-series equivalent to $F(s)$ consists of its vertical shifts. Further, as shown by Bohr, $f(s)$ is not equivalent to $F(s)$. On the other hand, $f(s)$ satisfies i) in Theorem 2.

References

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