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Analytic Lie extensions of number fields with cyclic fixed points and tame ramification

(Joint work with Farshid Hajir)

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1 Introduction

The conjecture of Fontaine and Mazur characterises all Galois representations which "come from algebraic geometry", that is, representations which arise as Tate twists of the action of G_K on subquotients of étale cohomology of some smooth projective varieties defined over K. The conjecture states that these representations are precisely the *geometric* representations, that is, representations which are unramified outside a finite set S of places v of K and which are *potentially semistable* at all places in S, that is, the restriction of ρ to the decomposition group at each place of K of residual characteristic p becomes semistable (in the sense of Fontaine). The *tame conjecture* of Fontaine and Mazur of ([3], conj 5a) considers only finitely and tamely ramified p-adic representations. Now, fix a prime $p > 3$. The conjecture is as follows

Conjecture 1.1 (Fontaine-Mazur) *Let K be a number field with absolute Galois group* $G_K = Gal(\bar{K}/K)$. *Let* $\rho : G_K \to GL_n(\mathbb{Q}_p)$ *be a continuous Galois representation such that*

1. *the representation is finitely ramified (that is, the set of ramified primes of* ρ *is finite).*
2. ρ *is unramified at p*

Then, the image of ρ *is finite.*

The philosophy of the conjecture: with the hypothesis of the nonramification at p, the eigenvalues of the Frobenius should be roots of unity. In this case, the image of ρ is solvable and by class field theory, the image is finite.

Definition 1.1 *A group* Γ *is uniform if and only if the following conditions are satisfied*

1. Γ *is a pro-p-group, that is, a projective limit of a finite p-group.*
2. *The commutator* $[\Gamma, \Gamma] \subseteq \Gamma^p$*; where* Γ^p *is the subgroup generated by the p-power of elements of* Γ.
3. Γ *is torsion-free.*

The first result in the direction of the conjecture is the following due to Boston [2]:

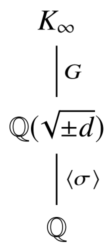
Theorem 1.1 (Boston) *Let K be a quadratic extension of k with Galois group* $\langle\sigma\rangle$. *Suppose that there is a uniform Galois extension L of K with Galois group* Γ *such that* L/K *is unramified and* L/k *is Galois. Suppose that the p-part of the classgroup of k is trivial, that is that p is relatively prime to the class number of k, then* Γ *is trivial.*

Thus, there is no arithmetic in such situation. Note that a uniform group is a special case of an analytic group. A p-adic analytic group is a closed subgroup of $GL_m(\mathbb{Z}_p)$ for some integer m. Lazard relates uniform groups and p-adic analytic groups in [1]:

Theorem 1.2 (Lazard) *Let G be a p-adic analytic pro-p group. Then G contains an open uniform subgroup.*

As G is compact, “open” means “of finite index”. For the conjecture of Fontaine-Mazur, one has to prove that something is finite, thus we can reduce to the case where the image of ρ is uniform.

The main ingredients of the proof of Boston. The element σ acts on Γ and since p is coprime to the class number of K, we have that σ does not act trivially on the abelianization $\Gamma^{ab} = \Gamma/[\Gamma, \Gamma]$. That is the action of σ on Γ^{ab} is fixed point free. As Γ is uniform, σ does not act trivially on Γ. Since σ has order 2 which is coprime to p and Γ is a pro-p group, we have that Γ is solvable and by class field theory, Γ is necessarily finite. By the definition of uniformity, Γ is torsion free. Hence, Γ is trivial.



This is not always the case: For example, consider the same situation. We know how to construct some extension where the Hilbert classfield tower is infinite. Here K_∞ is the p-Hilbert class field tower of K. Recall that the class group of $\mathbb{Q}$ is trivial. Now, σ does not act trivially on G^{ab} but the action of σ on G has some fixed points, that is, points $g \in G$ such that $\sigma(g) = g$ with $g \neq 1$. If there were no fixed points under the action of σ on G, the conclusion would be the same. That is, G should be solvable and then finite. But this is not the case. This is very particular to the uniform situation.

2 Uniform situation

What happens if we add some fixed points following the action of σ on Γ with Γ uniform. The context of Boston is “no fixed points”. So here we add some fixed points.

Examples of uniform groups The first uniform group that is non-trivial for the Fontaine-Mazur conjecture is

$$SL_2^1(\mathbb{Z}_p) = ker(SL_2(\mathbb{Z}_p) \to SL_2(\mathbb{F}_p)).$$

This group is uniform of dimension 3. More generally, the group $SL_n^1(\mathbb{Z}_p) = ker(SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{F}_p))$ is uniform of dimension $n^2 - 1$. We would like to obtain new uniform groups in the direction of the Fontaine-Mazur conjecture.

Class field towers. Let K be a number field and S a finite set of places of K. Let K_S be the maximal pro-p-extension of K unramified outside S, and $G_S = G_S(K) = Gal(K_S/K)$ be its Galois group. This extension is too big so we cut it. Let T be a finite set of places of K disjoint with S. Let K_S^T be the maximal extension of K such that there is no ramification outside S and every place in T splits totally. Put $G_S^T = Gal(K_S^T/K)$ be its Galois group.

We generalise the context of Boston, that is, look at the action of an element σ of order ℓ coprime to p. To simplify the exposition, take $\ell = 2$. We first define the concept of a σ-uniform image.

Definition 2.1 *Consider a continuous Galois representation*

$$\rho : G_S^T \to GL_n(\mathbb{Z}_p).$$

Let L be the subfield of K_S^T fixed by $\ker(\rho)$ *such that $\Gamma = im(\rho)$ is naturally identified with $Gal(L/K)$. Then, Γ is said to be σ-*uniform *if $\Gamma = Gal(L/K)$ is uniform and the extension L/k is Galois.*

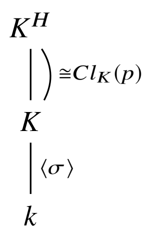
Theorem 2.1 (Hajir-Maire) *Let K be a quadratic extension of k. Suppose that s is a positive integer and that p does not divide the order of Cl_K. Let T be a set of primes of K sufficiently large, that is, the order of T satisfies $|T| \geq \alpha s + \beta$ with α, β constants depending on K. Then there exist s sets $S_1, \ldots, S_s$ of places of K, of positive (Chebotarev) density such that for every finite set $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ of places of K with $\mathfrak{p}_i \in S_i$, we have the following*

1. *The arithmetic is nontrivial. That is, $G_S^T(K)$ is infinite.*
2. *Under the action of σ on $(G_S^T)^{ab}$, there are s independent fixed points.*
3. *There is no continuous Galois representation $\rho : G_S^T \to GL_n(\mathbb{Z}_p)$ with σ-uniform image $SL_2^1(\mathbb{Z}_p)$ if:*
   a) *Fontaine-Mazur conjecture holds for the base field k.*
   or
   b) *$s \leq 2$ ("small").*

If we replace $SL_2^1(\mathbb{Z}_p)$ with $SL_n^1(\mathbb{Z}_p)$, the result still holds and we have $s \leq n^2 - 1$, if the action of σ on the group Γ corresponds to σ_A, conjugation by a matrix $A \in GL_n(\mathbb{Z}_p)$.

Sketch of proof. The first statement is a consequence of Golod-Shafarevich since $|T|$ grows linearly with s.

For the second statement, we need the following. Let K^H be the p-Hilbert class field of K, that is, the maximal abelian unramified p-extension of K. Let $Cl_K(p)$ be the p-sylow classgroup of K and $N = Gal(K^H/K)$. Then, Artin map gives the canonical isomorphism $Cl_K(p) \cong N$. The prime p divides the order of N, so we are not in the semisimple case.



We want to choose S in order to create enough fixed points for the action of σ. To find S we use Kummer theory and Chebotarev density theorem. To do this, we need to know more about the units $\mathcal{O}$ of K^H. The arithmetic question is to find a Minkowski unit from the extension K^H/K. Now, K^H has a Minkowski unit if $\mathcal{O}/(\mathcal{O})^p$ contains a nontrivial $\mathbb{F}_p[N]$-free module. We look at the structure of the units of K^H modulo p as an $\mathbb{F}_p[N]$-module. We introduce the set T and consider the T-units $\mathcal{O}^T$. We prove that when T is large, the T-units admit a large $\mathbb{F}_p[N]$-module.

Thus, we compare the Galois module structure coming from group theory by the action of σ on some subgroup of the analytic group, with the structure coming from arithmetic structure and by the choice of S, there is an incompatibility.

References

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