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Michel Waldschmidt

Continued Fractions: Introduction and Applications

written by Carlo Sanna

The continued fraction expansion of a real number x is a very efficient process for finding the best rational approximations of x. Moreover, continued fractions are a very versatile tool for solving problems related with movements involving two different periods. This situation occurs both in theoretical questions of number theory, complex analysis, dynamical systems... as well as in more practical questions related with calendars, gears, music... We will see some of these applications.

1 The algorithm of continued fractions

Given a real number x, there exist an unique integer $\lfloor x \rfloor$, called the *integral part* of x, and an unique real $\{x\} \in [0, 1[$, called the *fractional part* of x, such that

$$x = \lfloor x \rfloor + \{x\}.$$

If x is not an integer, then $\{x\} \neq 0$ and setting $x_1 := 1/\{x\}$ we have

$$x = \lfloor x \rfloor + \frac{1}{x_1}.$$

Again, if x_1 is not an integer, then $\{x_1\} \neq 0$ and setting $x_2 := 1/\{x_1\}$ we get

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}}.$$

This process stops if for some i it occurs $\{x_i\} = 0$, otherwise it continues forever. Writing $a_0 := \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \geq 1$, we obtain the so-called *continued fraction expansion* of x:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

which from now on we will write with the more succinct notation

$$x = [a_0, a_1, a_2, a_3, \ldots].$$

The integers $a_0, a_1, \ldots$ are called *partial quotients* of the continued fraction of x, while the rational numbers

$$\frac{p_k}{q_k} := [a_0, a_1, a_2, \ldots, a_k]$$

are called *convergents*. The convergents are the best rational approximations of x in the following sense: If p and $q > 0$ are integers such that

$$\left| \frac{p}{q} - x \right| < \frac{1}{2q^2}, \tag{1}$$

then p/q is a convergent of x. Indeed, of any two consecutive convergents p_k/q_k and p_{k+1}/q_{k+1} of x, one at least satisfies (1) (see [7, Theorems 183 and 184]).

If $x = a/b$ is a rational number, then the method for obtaining the continued fraction of x is nothing else than the Euclidean algorithm for computing the greatest common divisor of a and b:

$$\begin{array}{lll} a = a_0 b + r_0, & 0 \le r_0 < b, & x_1 = b/r_0, \\ b = a_1 r_0 + r_1, & 0 \le r_1 < r_0, & x_2 = r_0/r_1, \\ r_0 = a_2 r_1 + r_2, & 0 \le r_2 < r_1, & x_3 = r_1/r_2, \\ \dots & & \end{array}$$

Therefore, on the one hand, since the Euclidean algorithm always stops, the continued fraction of a rational number is always finite. On the other hand, it is obvious that a finite continued fraction represents a rational number. Hence, in conclusion, we have shown that a real number is rational if and only if its continued fraction expansion is finite.

Note that, if $a_k \ge 2$, then

$$[a_0, a_1, a_2, \dots, a_k] = [a_0, a_1, a_2, \dots, a_{k-1}, a_k - 1, 1], \qquad (2)$$

Thus a rational number can be expressed as a continued fraction in at least two ways. Indeed, it can be proved [7, Theorem 162] that any rational number can be written as a continued fraction in exactly two ways, which are given by (2).

2 The number of days in a year

Let us see an application of continued fractions to the design of a calendar. How many days are in a year? A good answer is 365. However, the astronomers tell us that the Earth completes its orbit around the Sun in approximately 365.2422 days. The continued fraction of this figure is

$$365.2422 = [365, 4, 7, 1, 3, 4, 1, 1, 1, 2].$$

The second convergent is

$$365.25 = 365 + \frac{1}{4},$$

which means a calendar of 365 days per year but a leap year every 4 years. The forth convergent gives the better approximation

$$365.2424\ldots = [365, 4, 7, 1] = 365 + \frac{8}{33}.$$

The Gregorian calendar, named after Pope Gregorio XIII who introduced it in 1582, is based on a cycle of 400 years: there is one leap year every year which is a multiple of 4 but not of 100 unless it is a multiple of 400. This means that in 400 years one omits 3 leap years, thus there are

$$400 \cdot 365 + 100 - 3 = 146097$$

days. On the other hand, in 400 years the number of days counted with an year of $365 + \frac{8}{33}$ days is

$$400 \cdot \left(365 + \frac{8}{33}\right) = 146096.9696\ldots$$

a very good approximation!

3 Design a planetarium

Christiaan Huygens (1629–1695) among being a mathematician, astronomer, physicist and probabilist, was also a great horologist. He designed more accurate clocks then the ones available at his time. In particular, his invention of the pendulum clock was a breakthrough in timekeeping. Huygens also built a mechanical model of the solar system. He wanted to design the gear ratios in order to produce a proper scaled version of the planetary orbits. He knew that the time required for the planet Saturn to orbit around the Sun is about

$$\frac{77708431}{2640858} = 29.425448\ldots = [29, 2, 2, 1, 5, 1, 4, \ldots].$$

The forth convergent is

$$[29, 2, 2, 1] = \frac{206}{7}.$$

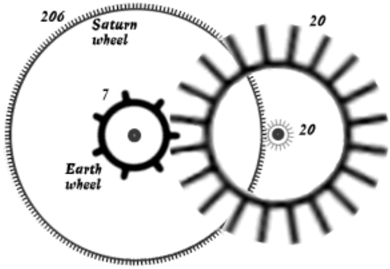

Figure 1: Huygens' planetary gears [1].

Therefore, Huygens made the gear regulating the Saturn's motion with 206 teeth, and the gear regulating Earth's motion with 7 teeth, as shown in Fig. 1.

4 Build a musical scale

The successive harmonics of a note of frequency n are the vibrations with frequencies $2n$, $3n$, $4n$, ... The successive octaves of a note of frequency n are the vibrations with frequencies $2n$, $4n$, $8n$, ... Our ears recognize notes at the octave one from another. Using octaves, one replaces each note by a note with frequency in a given interval, say $[n, 2n[$. The classical choice in Hertz is $[264, 528[$, which means tuning the C tone to 264 Hz (see [6, §20.3]). However, we shall use $[1, 2[$ for simplicity.

Hence, each note with frequency f is replaced by a note with frequency $r \in [1, 2[$ such that

$$f = 2^a r, \quad a = \lfloor \log_2 f \rfloor \in \mathbb{Z}, \quad r = 2^{\{\log_2 f\}} \in [1, 2[.$$

This is a multiplicative version of the Euclidean division.

A note with frequency 3, which is a harmonic of 1, is at the octave of a note of frequency $3/2$. The interval $[1, 3/2[$ is called *fifth*, and the ratio of its end points is $3/2$. The interval $[3/2, 2[$ is called *fourth*, with ratio $4/3$. The successive fifths are the notes in the interval $[1, 2[$ which are at the octave of notes with frequencies

$$1, 3, 9, 27, 81, \dots$$

namely:

$$1, \frac{3}{2}, \frac{9}{8}, \frac{27}{16}, \frac{81}{64}, \dots$$

We shall never come back to the initial value 1, since the Diophantine equation $2^a = 3^b$ has no solution in integers a and b.

In other words, the logarithm of 3 in basis 2 is irrational. Powers of 2 which are close to power of 3 correspond to good rational approximation a/b to $\log_2 3$. Thus it is natural to look at the continued fraction expansion:

$$\log_2 3 = 1.58496250072\dots = [1, 1, 1, 2, 2, 3, 1, 5, \dots].$$

The approximation

$$\log_2 3 \approx [1, 1, 1, 2] = \frac{8}{5}$$

means that $2^8 = 256$ is not too far from $3^5 = 243$, that is, 5 fifths produce almost 3 octaves. The next approximation

$$\log_2 3 \approx [1, 1, 1, 2, 2] = \frac{19}{12}$$

tells us that $2^{19} = 524288$ is close to $3^{12} = 531441$, that is

$$\left(\frac{3}{2}\right)^{12} = 129.74\dots \approx 2^7 = 128.$$

This means that 12 fifths are just a bit more than 7 octaves.

The figure

$$\frac{3^{12}}{2^{19}} = 1.01364,$$

is called the *Pythagorean comma* (or ditonic comma) and produces an error of about 1.36‰, which most people cannot hear.

Further remarkable approximations between perfect powers are:

$$5^3 = 125 \approx 2^7 = 128,$$

that is,

$$\left(\frac{5}{4}\right)^3 = 1.953\ldots \approx 2,$$

so that 3 thirds (ratio 5/4) produce almost 1 octave; and

$$2^{10} = 1024 \approx 10^3,$$

which means that one kibibyte (1024 bytes) is about one kilobyte (1000 bytes), and that doubling the intensity of a sound is close to adding 3 decibels.

5 Exponential Diophantine equations

Another way to avoid the problem that we cannot solve the equation $2^a = 3^b$ in positive integers a and b, might be looking for powers of 2 which are just one unit from powers of 3, that is $|2^a - 3^b| = 1$. This question was asked by the French composer Philippe de Vitry (1291–1361) to the medieval Jewish philosopher and astronomer Levi ben Gershon (1288–1344). Gershon proved that there are only three solutions (a, b) to the Diophantine equation $2^a - 3^b = \pm 1$, namely $(1, 1)$, $(2, 1)$, $(3, 2)$.

Indeed, suppose that $2^a - 3^b = -1$. If $a = 1$ then, obviously, $b = 1$. If $a \geq 2$ then $3^b \equiv 1 \pmod 4$, so that $b = 2k$ for some positive integer k, and consequently

$$2^a = 3^b - 1 = (3^k - 1)(3^k + 1),$$

which implies that both $3^k - 1$ and $3^k + 1$ are powers of 2. But the only powers of 2 which differ by 2 are 2 and 4, hence $k = 1$, $b = 2$, and $a = 3$.

Similarly, suppose that $2^a - 3^b = 1$. Hence $2^a \equiv 1 \pmod 3$, so that $a = 2k$ for some positive integer k and

$$3^b = 2^a - 1 = (2^k - 1)(2^k + 1),$$

which implies that both $2^k - 1$ and $2^k + 1$ are powers of 3. But the only powers of 3 which differ by 2 are 1 and 3, hence $k = 1$, $a = 2$, and $b = 1$.

This kind of questions lead to the study of the so called *exponential Diophantine equations*. A notable case is the *Catalan's equation*

$$x^p - y^q = 1,$$

where x, y, p, q are integers all ≥ 2. In 2002 Mihăilescu [9] showed that $3^2 - 2^3 = 1$ is the only solution, as conjectured by Catalan in 1844.

6 Electric networks

The electrical resistance of a series of two resistances R_1 and R_2 is $R_1 + R_2$ (see Fig. 2). If R_1 and R_2 are instead in a parallel network (see

Figure 2: Two resistances R_1 and R_2 in series.

Fig. 3), then the resulting resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Therefore, it follows easily that the resistance U of the circuit of Fig. 4

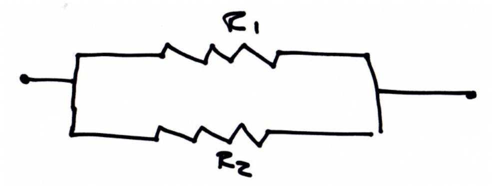

Figure 3: Two resistances R_1 and R_2 in parallel.

is given by

$$U = \cfrac{1}{S + \cfrac{1}{R + \cfrac{1}{T}}}.$$

A similar kind of reasoning shows that the resistance of the infinite

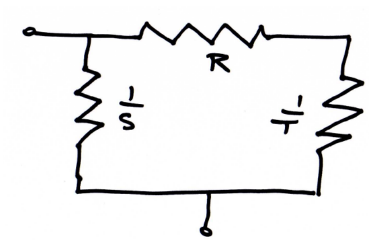

Figure 4: A series–parallel network.

circuit of Fig. 5 is given by the following continued fraction expansion

$$[R_0, S_1, R_1, S_2, R_2, \ldots].$$

Electric networks and continued fractions have been used to solve the "Squaring the square" problem, which states: Is it possible to decompose an integer square into the disjoint union of integer squares, all of which are distinct? The answer to this problem is positive. Indeed, in 1978 Duijvestijn found a decomposition of the 122×122

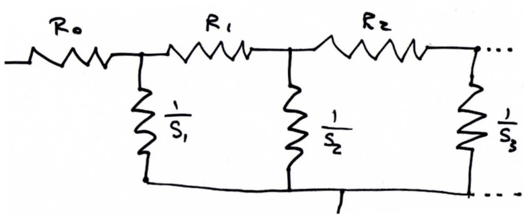

Figure 5: An infinite circuit.

square into 21 distinct integer squares (see Fig. 6). Furthermore, there are no solutions with less than 21 squares, and Duijvestijn's solution is the only with 21 squares (see [3]).

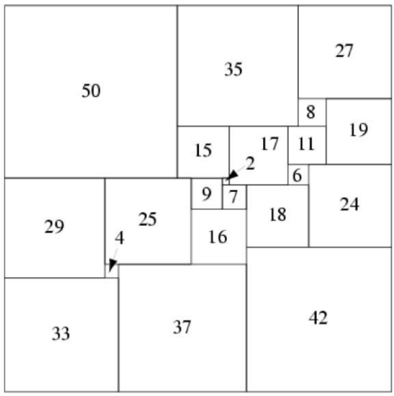

Figure 6: Duijvestijn's solution.

7 Quadratic numbers

Joseph-Louis Lagrange (1736–1813) proved that the continued fraction expansion of a real number x is ultimately periodic, i.e.,

$$x = [a_0, \dots, a_k, b_1, \dots, b_h, b_1, \dots, b_h, \dots]$$

if and only if x is a quadratic number, that is, x is the root of a quadratic polynomial with rational coefficients (see [5, Chap. IV, §10]).

In such a case, we use the shorter notation

$$x = [a_0, \dots, a_k, \overline{b_1, \dots, b_h}],$$

in a ways similar to how it is done for repeating decimals.

7.1 Fibonacci sequence and the Golden Ratio

The Fibonacci sequence $(F_n)_{n \geq 0}$ was introduced by Leonardo Pisano (1170–1250), also known as *Fibonacci*. It is defined as $F_0 := 0$, $F_1 := 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$, and its first terms are

$$0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad \dots$$

The unique positive real numbers Φ satisfying

$$\Phi = 1 + \frac{1}{\Phi} \tag{3}$$

is given by

$$\Phi = \frac{1 + \sqrt{5}}{2}$$

and it is known as the *Golden Ratio*. The Golden Ratio makes its appearance in many different contexts, from Mathematics to Arts [8].

From (3) it is clear that the continued fraction expansion of Φ is

$$\Phi = [1, 1, 1, \dots] = [\overline{1}],$$

the simplest infinite continued fraction. Notably, the convergents of Φ are precisely the ratios of consecutive Fibonacci numbers

$$[1] = \frac{F_2}{F_1}, \quad [1,1] = \frac{F_3}{F_2}, \quad [1,1,1] = \frac{F_4}{F_3}, \quad [1,1,1,1] = \frac{F_5}{F_4}, \quad \ldots$$

so that

$$\Phi = \lim_{n \to +\infty} \frac{F_{n+1}}{F_n}.$$

7.2 Continued fraction for $\sqrt{2}$

The square root of 2 satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1},$$

while

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}},$$

hence the continued fraction expansion of $\sqrt{2}$ is given by

$$\sqrt{2} = [1,2,2,2,\ldots] = [1,\overline{2}].$$

7.3 Paper folding

The number $\sqrt{2}$ appears in the A series paper sizes. Precisely, since $\sqrt{2}$ is twice its inverse, i.e., $\sqrt{2} = 2/\sqrt{2}$, folding a rectangular piece of paper with sides in proportion $\sqrt{2}$ yields a new rectangular piece of paper with sides in proportion $\sqrt{2}$ again. The sizes of an A0 paper are defined to be in proportion $\sqrt{2}$ and so that the area is 1 m^2. Thus, rounded to the nearest millimetre, an A0 paper is 841 by 1189 millimetres. Note that

$$\frac{841}{1189} = \frac{29}{49} = [1,2,2,2,2]$$

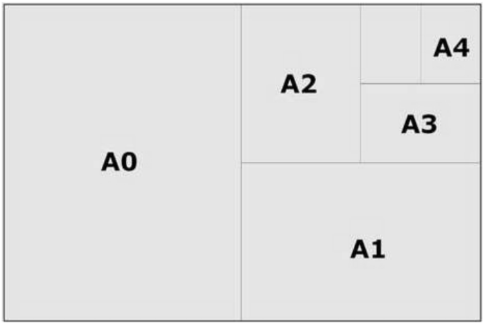

Figure 7: The A series format.

is the fifth convergent of $\sqrt{2}$. The sizes of A1, A2, A3, and so forth are defined by successively halving the A0 paper, as in Fig. 7.

The Golden Ratio Φ has a similar property. If we start with a rectangle with Golden Ratio proportion, then we can fold it in order to get a square and a smaller rectangle which sizes are again in Golden Ratio proportion, as shown in Fig. 8. In fact, the Golden Ratio is the

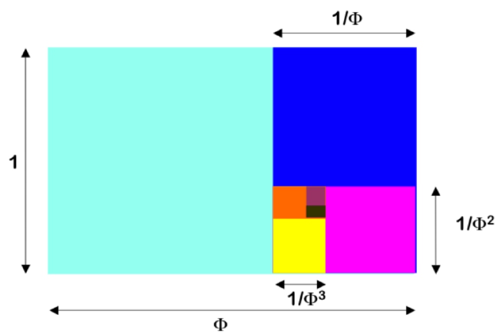

Figure 8: The "Golden rectangle".

unique number with this property.

7.4 The irrationality of $\sqrt{2}$: Two geometric proofs

Considerations similar to the ones of the previous section can lead to "geometric" proofs of the irrationality of $\sqrt{2}$.

A first proof is the following:

- Start with a rectangle having side lengths 1 and $1 + \sqrt{2}$ (see Fig. 9).
- Decompose it into two squares of sides 1 and a rectangle of sides 1 and $1 + \sqrt{2} - 2 = \sqrt{2} - 1$.
- The second rectangle has sides in proportion

  $$\frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2},$$

  hence it can be decomposed in two squares and a rectangle whose sides are again in $1 + \sqrt{2}$ proportion.
- This process does not end.

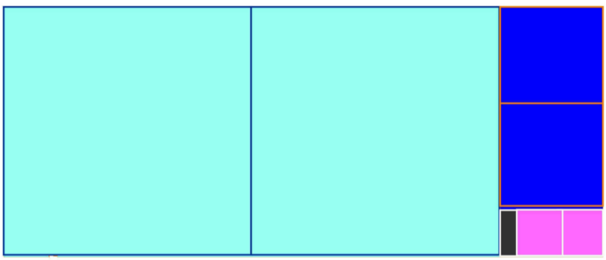

Figure 9: A rectangle dissection proving the irrationality of $\sqrt{2}$.

If we were started with a rectangle having integer side lengths, then it is clear that the process would have stopped after finitely many steps (the side lengths of the successive rectangles produce a decreasing sequence of positive integers). The same conclusion holds for a rectangle with side lengths in rational proportion (reduce to a common denominator and scale). Therefore, $1 + \sqrt{2}$ is irrational, and so is $\sqrt{2}$.

It is also possible to give a proof in just one dimension:

- Start with an interval of length $t = 1 + \sqrt{2}$ (see Fig. 10).
- Decompose it in two intervals of length 1 and one interval of length $\sqrt{2} - 1 = 1/t$.
- The smaller interval can now be split in two intervals of length $1/t^2$ and one of length $1/t^3$.
- This process does not stop.

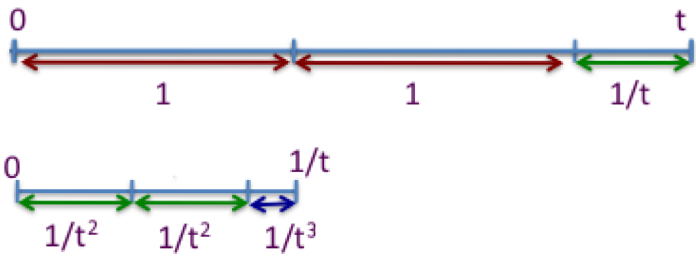

Figure 10: An interval dissection proving the irrationality of $\sqrt{2}$.

Reasoning in a way similar to the previous, it follows easily that if the interval length is a rational number then the process must stop. Thus we get again that $\sqrt{2}$ is irrational.

7.5 The Pell's equation $x^2 - dy^2 = 1$

Let d be a positive integer which is not a square. The Diophantine equation

$$x^2 - dy^2 = 1 \tag{4}$$

is known as *Pell's equation* [5, Chap. IV, §11]. It can be rewritten as

$$\left(x - \sqrt{d}y\right)\left(x + \sqrt{d}y\right) = 1,$$

hence, for $y > 0$, we have that x/y is a rational approximation of $\sqrt{d}$. This is the reason why a strategy for solving (4) is based on the continued fraction expansion of $\sqrt{d}$.

It is quite curious that for relatively small values of d the solutions (x, y) of (4) can be very large. For example, the Indian mathematician Brahmagupta (~628) asked for solution for $d = 92$. The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9, \overline{1, 1, 2, 4, 2, 1, 1, 18}],$$

and a solution $(x, y) = (1151, 120)$ is obtained from

$$[9, 1, 1, 2, 4, 2, 1, 1] = \frac{1151}{120}.$$

Another example is the one of Bhaskara (~1150), that using the same method of Brahmagupta showed that a solution for $d = 61$ is given by

$$x = 1766319049, \quad y = 226153980.$$

But a more impressive example was given by Fermat, who asked to his friend Brouncker a solution for $d = 109$, saying that he choose a small value of d to make the problem not too difficult. However, the smallest solution is

$$x = 158070671986249, \quad y = 15140424455100,$$

which is also given by

$$\left(\frac{261 + 25\sqrt{109}}{2}\right)^6 = 158070671986249 + 15140424455100\sqrt{109}.$$

8 Continued fractions for e and π

Leonard Euler (1707–1784) proved that the continued fraction for e is given by

$$\begin{aligned} e &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots] \\ &= [2, \overline{1, 2m, 1}]_{m \geq 1}. \end{aligned}$$

This result implies that e is not rational neither a quadratic irrational. (Indeed, in 1874 Charles Hermite proved that e is transcendental.) Actually, Euler showed the more general result that for any integer $a \geq 1$ it holds

$$\begin{aligned} e^{1/a} &= [1, a-1, 1, 3a-1, 1, 1, 5a-1, 1, \ldots] \\ &= [\overline{1, (2m+1)a-1, 1}]_{m \geq 1}. \end{aligned}$$

Johann Heinrich Lambert (1728–1777) proved $\tan(v)$ is irrational when $v \neq 0$ is rational. Hence π is irrational, since $\tan(\pi/4) = 1$. The continued fraction expansion of π,

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, \ldots].$$

is much more mysterious than the one of e. Indeed, it is still an open problem to understand if the sequence of partial quotients of π is bounded or not.

9 Continued fractions for analytic functions

Also some analytic functions have a kind of continued fraction expansion. For example, the tangent:

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}.$$

The study of continued fractions of analytic functions is strictly connected to the theory of Padé approximations, which are rational function approximations of analytic functions (see [4]).

10 Gauss map and ergodic theory

Let (X, μ) be a probability space and let $H : X \to X$ be a map that preserve the measure μ, i.e., $\mu(H^{-1}(E)) = \mu(E)$ for any measurable $E \subseteq X$. The Birkhoff's Ergodic Theorem [13, §1.6] states that if H is *ergodic*, which means that $H^{-1}(E) = E$ implies $\mu(E) = 0$ or $\mu(E) = 1$, then for any $f \in L^1_\mu(X)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(H^{(k)}(x)) = \int_X f \mathrm{d}\mu,$$

for almost all $x \in X$, respect to the measure μ, where $H^{(k)}$ denotes the k-th iterate of H.

We have seen that the partial quotient of a continued fraction are obtained by iterating the map

$$T : x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

which is called *Gauss map*. It can be proved that the Gauss map preserve the measure

$$\mu(E) := \frac{1}{\log 2} \int_E \frac{\mathrm{d}x}{x+1}, \quad E \subseteq [0, 1],$$

and that it is ergodic. This facts connect continued fractions with the study of chaotic dynamical systems. In particular, exploiting this connection, it can be proved the following result of Aleksandr Yakovlevich Khinchin: For all real numbers

$$x = [a_0, a_1, a_2, \ldots],$$

but a set of Lebesgue measure zero, it holds

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} a_k} = K_0,$$

where

$$K_0 := \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\log_2 r} \approx 2.685452\ldots$$

is known as *Khinchin's constant*.

11 Connection with the Riemann zeta function

We recall that for real $s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Notably, $\zeta(s)$ is related to the Gauss map T by the following formula

$$\zeta(s) = \frac{1}{s-1} - s \int_0^1 T(x) x^{s-1} \mathrm{d}s.$$

12 Generalizations of continued expansion in higher dimension

Simultaneous rational approximations of real numbers is a much more difficult problem than the rational approximation of a single number. In fact, the continued fraction expansion algorithm has many specific features and so far there is no extension of this algorithm in higher dimension with all such properties.

However, some attempts has been made, in particular the Jacobi–Perron algorithm [2] uses a kind of ternary continued fraction expansion to deal with cubic irrationality. This topic is strictly related the *Geometry of numbers*, started by Hermann Minkowski (1864–1909), which is the study of convex bodies and integer vectors in the n-dimensional space $\mathbb{R}^n$. One of the most important result of this field is the LLL algorithm [10], named after Arjen Lenstra, Hendrik Lenstra and Laszlo Lovasz, that given m vectors in $\mathbb{R}^n$ it produces a basis of the lattice they generate with often a smaller norm than the initial ones.

For more recent results see the works of Wolfgang Schmidt, Leonhard Summerer, and Damien Roy [11, 12] in the so-called *Parametric geometry of numbers*.

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Carlo Sanna
Department of Mathematics "Giuseppe Peano"
Università degli Studi di Torino
Via Carlo Alberto 10
10123 Torino, Italy.
email: carlo.sanna.dev@gmail.com