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Limiting values of Lambert series and the secant zeta-function

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The functional equation for the Riemann zeta-function $\zeta(s)$, i.e.

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

can be derived noticing that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^{\infty}x^{\frac{1}{2}s-1}\omega(x)dx,$$

where the function $\omega(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$ is trivially related to the theta-function $\theta(x)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}$ for which the following theta-transformation formula holds:

$$\theta(x)=\frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right).$$

A similar approach can be adapted to obtain the Davenport-Chowla identity [2], [3], [4],

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\psi(nx)=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi n^2 x}{n^2}, \tag{1}$$

where $\psi(x) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n}$ is the saw-tooth Fourier series and $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function, whose associated Dirichlet series is

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \qquad \sigma = \Re(s) > 1\,.$$

In (1), on the one hand, there appears $\lambda(n)$ which is a prime number-theoretic entity and, on the other hand, a Riemann's example of a nowhere differentiable function, $\psi(x)$. The integrated identity can be derived from the functional equation only, but to differentiate it one needs the estimate for the error term for the Liouville function which is as deep as the PNT:

$$\sum_{n\le x}\lambda(n) = O(xe^{-c\log^{3/5}x(\log\log x)^{-1/5}})\,.$$

The right-hand side of (1) may be viewed as the imaginary part of the integrated theta-series, so the theta-transformation formula and the functional equation are equivalent. It seems that the uniform convergence of the left-side and the differentiability of the right-side merge as the limiting behavior of a sort of modular function and the Riemann zeta-function, which is modular-function-related.

To establish the Davenport-Chowla identity (1), we need to prove the integrated form by the functional equation and then differentiate. In order to establish an identity in general, we are to integrate it and then differentiate the resulting integral form (or differencing) to deduce it: this is the Abel-Tauber process. It is best known when applied to series. The Riesz sum and its differencing, proving the integrated identity and then differentiating it to obtain the desired identity, the radial integration and radial limits etc. may all be thought of as an Abel-Tauber process. We recall Perron's formula

$$\frac{1}{\Gamma(\varkappa+1)}\sum_{\lambda_k\le x}{}' \alpha_k(x-\lambda_n)^{\varkappa} = \frac{1}{2\pi i}\int_c \frac{\Gamma(s)\varphi(s)x^{s+\varkappa}}{\Gamma(s+\varkappa+1)}\,\mathrm{d}s,$$

where the left-hand side sum is called the Riesz sum of order $\varkappa$ and $\varphi(s) = \sum_{k=1}^{\infty} \frac{\alpha_k}{k^s}$, see [6].

Defining $\Theta(z) = \theta(-iz) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$, the theta-transformation formula now reads

$$\Theta(z) = e^{\frac{\pi i}{4}} z^{-\frac{1}{2}} \Theta\left(-\frac{1}{z}\right) .$$

Then $F(z) = \sum_{n=1}^{\infty} \frac{e^{\pi i n^2 z}}{\pi i n^2}$ is essentially the integral of $\Theta(z)$, since

$$\int_0^z \Theta(z)\, dz = z + 2\left(\sum_{n=1}^{\infty} \frac{e^{\pi i n^2 z}}{\pi i n^2} - \sum_{n=1}^{\infty} \frac{1}{\pi i n^2}\right) = z + 2(F(z) - F(0)) .$$

From [1] we have the following:

Theorem 1.

$$F\left(\frac{2q}{p} + \mathfrak{z}\right) - F\left(\frac{2q}{p} + i\epsilon\right) = S(p,q)\frac{e^{-\pi i/4}}{p}\sqrt{\mathfrak{z}} - \frac{1}{2}h + O(\mathfrak{z}^2)$$

where $S(p,q)$ indicates the quadratic Gauss sum defined for $b \in \mathbb{N}$ by $S(b,a) = \sum_{j=0}^{b-1} e^{2\pi i j^2 \frac{a}{b}}$.

The classical Gauss' quadratic reciprocity law claims whether p is a quadratic residue or non-residue modulo q once q is a quadratic residue or non-residue modulo p is known: but this is not a priori clear. It seems that this is one of the avatars of the symmetry associated with the zeta-functions, i.e. with the functional equation. In our case, we generalize to the following:

Theorem 2.

$$S(p,q) = e^{\frac{\pi}{4}\operatorname{sgn}(q) i}\left(\frac{p}{2|q|}\right)^{1/2} S(4|q|, -\operatorname{sgn}(q)p) .$$

Corollary 1.

$$F\left(\frac{q}{p}+\mathfrak{z}\right)-F\left(\frac{q}{p}+i\epsilon\right)=R(p,q)\frac{e^{-\pi i/4}}{p}\sqrt{\mathfrak{z}}-\frac{1}{2}h+O(\mathfrak{z}^2),$$

where

$$\begin{aligned}
&R(p,2q)=S(p,q)=\varepsilon(p)\left(\frac{q}{p}\right)\sqrt{p},\\
&R(2p,q)=S(4p,q)=e^{\frac{\pi}{4}i}\sqrt{2p}\left(\frac{-p}{q}\right),\\
&R(2B+1,2A+1)=0\,.
\end{aligned}$$

There are many generalizations of the Dedekind eta-function as a Lambert series. Lerch [8] in 1904 introduced the cotangent zeta-function for algebraic irrational z and odd positive integers s as

$$\xi(z,s)=\sum_{n=1}^{\infty}\frac{\cot(n\pi z)}{n^s}\,.$$

Recently, Lalín et al. [7] considered the secant zeta function

$$\psi(z,s)=\sum_{n=1}^{\infty}\frac{\sec(n\pi z)}{n^s}$$

and found its special values at some particular quadratic irrational arguments. The main result of Lalín et al [7, Theorem 3] concerns the difference

$$\begin{aligned}
&(\alpha+1)^{l-1}\psi\left(\frac{\alpha}{\alpha+1},l\right)-(-\alpha+1)^{l-1}\psi\left(\frac{\alpha}{-\alpha+1},l\right)\\
=&\frac{(\pi i)^l}{l!}\sum_{m=0}^{l}(2^{m-1}-1)B_mE_{l-m}\binom{l}{m}\left[(1+\alpha)^{m-1}-(1-\alpha)^{m-1}\right]
\end{aligned}\tag{2}$$

which can be expressed in terms of Bernoulli and Euler numbers.

In [5] we generalized those results. Defining

$$A^*\left(\alpha, s, \frac{1}{2}, 0\right) = \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\frac{1}{\cos \pi k\alpha} = \frac{1}{2}\psi(\alpha, 1-s)$$

and

$$V_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V_2 = V_0^2 V_1^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

consider the difference

$$D^*(V) = D^*\left(V\alpha, s, \left(\frac{1}{2}, 0\right)\right) = A^*\left(V\alpha, s, \left(\frac{1}{2}, 0\right)\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right)$$

for each V. We have the following:

Theorem 3.

$$(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1}, s, \left(\frac{1}{2}, 0\right)\right) + (\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1}, s, \left(\frac{1}{2}, 0\right)\right)$$
$$= \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{\left(1 - e\left[\frac{s}{2}\right]\right)} \int_{I(\lambda,\infty)} t^{s-1} \sum_{m=0}^{\infty} 2^{-m-1} E_m \frac{t^m}{m!} \sum_{n=0}^{\infty} (2^{1-n} - 1) B_n$$
$$\times \frac{\{(\alpha+1)^{n-1} + (\alpha-1)^{n-1}\} t^{n-1}}{n!}\, dt\,.$$

This Theorem involves the sum of $D^*(V_1)$ and $D^*(V_2)$, which is the genesis of the transformation formula of Lalin et al. [7, Theorem 3], (2), for the secant zeta function. Differently from (2), the result is to be the sum rather than the difference. The oddness of the integer $l-1$ gives a disguised form to the formula. As can be seen in the proof given in the paper [5], $2A^*\left(\alpha, s, \frac{1}{2}, 0\right)$ on the left side and the sum of secant zeta-functions on the right naturally cancel each other. Since this occurs only in such a pairing, this elucidates the hidden structure of the paired transformation formula from a more general standpoint.

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