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Relations among arithmetical functions, sum of digits functions and paper-folding sequences

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1 Introduction

In the field of arithmetical function, we sometimes come across an interesting phenomenon, for example, the difference function of "the sum of digits function for Reflected Binary Code (RBC)", $\{S_{RBC}(n) - S_{RBC}(n-1)\}_{n=1}^{\infty}$, coincides with the regular paper-folding sequence. Here we talk about a generalization of this phenomenon and describe about some relations among the sum of digits functions, automatic sequences and some code systems. This is a survey of the papers [2] and [3], and as for the proofs please refer to these references.

2 An example

2.1 Reflected Binary Code (RBC)

We set

- S_{RBC} is the sum of digits function for RBC

- $\Delta(n) := S_{RBC}(n) - S_{RBC}(n-1)$, for any positive integer n.

Then

n	Usual Binary Code	RBC	$S_{RBC}(n)$	$\Delta(n)$
0	0	0	0	
1	1	1	1	+1
2	10	11	2	+1
3	11	10	1	−1
4	100	110	2	+1
5	101	111	3	+1
6	110	101	2	−1
7	111	100	1	−1
8	1000	1100	2	+1
9	1001	1101	3	+1
10	1010	1111	4	+1
⋮	⋮	⋮	⋮	⋮

We would like to point out that

⊙ RBC is a permutation of Binary Code (BC).

⊙ RBC is a "Gray Code ": RBC for $n+1$ and RBC for n differ by exactly one digit. Thus

$$\forall n \in \mathbb{N}, |\Delta(n)| = 1$$

2.2 Regular paper-folding sequence

We fold a paper to the same direction (counter-clockwise).Then we get folds ”$V-type$” or ”$\Lambda-type$” progressively. Mathematically, we start from the simple sequence

$$\mathfrak{b}_0 = \{1, 1, 1, 1, \cdots\}$$

with $b_n = 1, \forall n \in \mathbb{N}$ and then we construct the sequence $\mathcal{P}_{\mathfrak{b}_0}$, which we call the regular paper-folding sequence as follows

$$\mathcal{P}_{\mathfrak{b}_0} = \{\mathbf{1}, \mathbf{1}, -1, \mathbf{1}, 1, -1, -1, \mathbf{1}\}$$

Thus one has

Theorem 1 ([2]).

$$\forall n \in \mathbb{N}, \Delta(n) = \mathcal{P}_{\mathfrak{b}_0}(n).$$

More generally, if

$$\mathfrak{b} := \{b_1, b_2, b_3, \cdots\} \text{ with } b_1 = 1 \text{ and } b_i = \pm 1 \text{ for } i \geq 2$$

Then, as before, we get the generalized paper-folding sequence $\mathcal{P}_{\mathfrak{b}}$

$$\mathcal{P}_{\mathfrak{b}} = \{\mathbf{b_1}, \mathbf{b_2}, -b_1, \mathbf{b_3}, b_1, -b_2, -b_2, \mathbf{b_4}, \cdots\}$$

Thus it is natural to ask ourselves

Question 2. *Is there a corresponding code* C *such that*

$$\{S_C(n) - S_C(n-1)\}_{n=1}^{\infty} = \{\mathcal{P}_{\mathfrak{b}}(n)\}_{n=1}^{\infty} \ ?$$

The answer is given below.

3 Arithmetical function and sum of digits function

Let go back to the RBC and let define the function

$$\begin{aligned} \xi_{RBC} : \mathbb{R}_+ &\to \mathbb{N} \cup \{\infty\} \\ x &\mapsto \sum_{0 \leq n \leq x} \chi_4(n) \end{aligned}$$

where χ_4 is the Dirichlet character modulo 4:

$$\chi_4 : \mathbb{N} \cup \{\infty\} \to \mathbb{C} \text{ such that } \chi_4(n) = \begin{cases} 0 & \text{if } n \equiv 0, 2 \mod 4 \\ 1 & \text{if } n \equiv 1 \mod 4 \\ -1 & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Then one has ([2])

$$S_{RBC}(n) = \sum_{k=0}^{\infty} \xi_{RBC}\left(\frac{n}{2^k}\right).$$

This expression could be useful in the study of sum of digits functions.

Remark 3. *For the case of BC, setting*

$$f(n) := \begin{cases} 0 & \text{if } n = 0 \\ (-1)^{n-1} & \text{if } n \geq 1 \end{cases}$$

And

$$\xi_{BC}(x) := \sum_{0 \leq n \leq x} f(n)$$

Then one has ([2])

$$S_{BC}(n) = \sum_{k=0}^{\infty} \xi_{BC}\left(\frac{n}{2^k}\right).$$

From this expression, one can derive the excellent result on the average of S_{BC} due to H. Delange (1975).

In order to answer question 2, let give a structure of RBC

In RBC $\boxed{0110}$ is the "unit" and this unit comes from $\xi_{RBC}(x)$.

$$\begin{array}{llll} \boxed{0110}\,\boxed{0110} & \boxed{0110}\,\boxed{0110} & \boxed{\varpi}\,\boxed{\varpi} & \cdots \\ \boxed{0011\ \ 1100} & \boxed{0011\ \ 1100} & \boxed{\ \ 2\varpi\ \ } & \cdots \\ \multicolumn{2}{l}{\boxed{0000\ \ 1111\quad 1111\ \ 0000}} & \boxed{\qquad 4\varpi \qquad} & \\ & \vdots & & \end{array}$$

We repeat $\boxed{\varpi}$, $\boxed{2\varpi}$, $\boxed{4\varpi}$, $\cdots$ as above and then, reading vertically, we get the code RBC(n).

Taking into account of the generalization of $\mathfrak{b}_0$ to $\mathfrak{b}$, we can construct by the same way a new code $C_{\mathfrak{b}}$ as follows:
If

$$\mathfrak{b} = \{b_1, b_2, b_3, \cdots\} \text{ with } b_1 = 1 \text{ and } b_i = \pm 1 \text{ for } i \geq 2$$

Then $C_{\mathfrak{b}}$ is (reading vertically)

$$\boxed{0b_1b_10}\,\boxed{0b_1b_10}\quad\boxed{0b_1b_10}\,\boxed{0b_1b_10}\quad\boxed{\varpi}\,\boxed{\varpi}\quad\cdots$$

$$\boxed{00b_2b_2\quad b_2b_200}\quad\boxed{00b_2b_2\quad b_2b_200}\quad\boxed{b_2\times 2\varpi}\quad\cdots$$

$$\boxed{0000\quad b_3b_3b_3b_3\quad b_3b_3b_3b_3\quad 0000}\quad\boxed{b_3\times 4\varpi}$$

$$\vdots$$

Thus one can show that $C_{\mathfrak{b}}$ is a Gray code.
Moreover we have:

Theorem 4 ([3])**.**

$$\{S_{C_{\mathfrak{b}}}(n) - S_{C_{\mathfrak{b}}}(n-1)\}_{n=1}^{\infty} = \{\mathcal{P}_{\mathfrak{b}}(n)\}_{n=1}^{\infty}.$$

We remark that $\{\mathcal{P}_{\mathfrak{b}}(n)\}_{n=1}^{\infty}$ is an example of automatic sequences.

4 Some properties of the code $C_{\mathfrak{b}}$

In the case of BC:

$$BC(13) = \underset{(\longrightarrow)}{1011}$$

Which means

$$1\cdot 2^0 + 0\cdot 2^1 + 1\cdot 2^2 + 1\cdot 2^3 = 13$$

i.e.

$$BC(n) = \underset{(\longrightarrow)}{\beta_1\beta_2\beta_3\cdots}$$

if and only if

$$\sum_{k=1}^{\infty}\beta_k 2^{k-1} = n$$

Moreover, this relation give a bijection between $\{BC(n)\}_{n=1}^{\infty}$ and $\mathbb{N}$.
We can prove a similar result for $C_{\mathfrak{b}}$. For this purpose, let define the function:

$$\mathcal{D}_{\mathfrak{b}} : \mathbb{N} \to \mathbb{Z}$$

Such that if

$$\mathcal{C}_{\mathfrak{b}}(n) = \underset{(\longrightarrow)}{\alpha_1\alpha_2\alpha_3\cdots}$$

Then

$$\mathcal{D}_{\mathfrak{b}}(n) := \sum_{k=1}^{\infty} \alpha_k 2^{k-1}.$$

Thus we have the following result

Theorem 5 ([3])**.** *Let $K \geq 2$ be an integer.*
If $\mathfrak{b}$ is $K-$periodic then the function $\mathcal{D}_{\mathfrak{b}}$ is a bijection between $\mathbb{N} \cup \{0\}$ and $\mathbb{Z}$.

Remark 6. *If $K = 1$ then $\mathfrak{b} = \mathfrak{b}_0$ and in this case $\mathcal{D}_{\mathfrak{b}_0}$ is a bijection from $\mathbb{N} \cup \{0\}$ onto itself.*

In 2, we remarked that:

$$f(n)=\begin{cases} 0 & \text{if } n = 0 \\ (-1)^{n-1} & \text{if } n \geq 1 \end{cases} \rightsquigarrow \xi_{BC}(x) \rightsquigarrow \sum_{k=0}^{\infty} \xi_{BC}\left(\frac{n}{2^k}\right) = S_{BC}(n)$$

$$\chi_4(n) \rightsquigarrow \xi_{RBC}(x) \rightsquigarrow \sum_{k=0}^{\infty} \xi_{RBC}\left(\frac{n}{2^k}\right) = S_{RBC}(n)$$

Then

Question 7. *What is the arithmetical function $g(n)$ which induces $g(n) \rightsquigarrow \xi_g(x) \rightsquigarrow \sum_{k=0}^{\infty} \xi_g(n/2^k) = S_{\mathcal{C}_{\mathfrak{b}}}(n)$?*

To this question, we have the answer which is as follows.
Let $p \geq 2$ and

$\mathcal{A} = \{g : \mathbb{N} \cup \{\infty\} \to \mathbb{C} \text{ with } g(0) = 0\}$ a set of arithmetical functions.

For $g \in \mathcal{A}$, we define the maps

$$\xi_g : \mathbb{R}_+ \to \mathbb{C}, x \mapsto \sum_{0 \leq n \leq x} g(n).$$

And then, we define the maps Φ_p and Ψ_p as follows: $\forall (g, n) \in \mathcal{A} \times \mathbb{N}$,

$$\Phi_p(g)(n) = \sum_{k=0}^{\infty} \xi_g \left(\frac{n}{p^k} \right)$$

$$\Psi_p(g)(n) = \begin{cases} 0 & \text{if } n = 0 \\ g(n) - g(n-1) - \left(g\left(\frac{n}{p}\right) - g\left(\frac{n}{p} - 1\right) \right) & \text{if } n \geq p \text{ and } n \equiv 0 \mod p \\ g(n) - g(n-1) & \text{otherwise} \end{cases}$$

Then we have:

Theorem 8 (Kamiya-Murata [2]). *Φ_p and Ψ_p are bijections from $\mathcal{A}$ onto itself and $\Phi_p^{-1} = \Psi_p$.*

This implies in particular:

$$f \underset{\Psi_2}{\overset{\Phi_2}{\rightleftharpoons}} S_{BC}$$

$$\chi_4 \underset{\Psi_2}{\overset{\Phi_2}{\rightleftharpoons}} S_{RBC}$$

And assuming $\mathfrak{b}$ $K-$periodic then $\exists f_{\mathfrak{b}} \in \mathcal{A}$ such that

$$f_{\mathfrak{b}} \underset{\Psi_{2K}}{\overset{\Phi_{2K}}{\rightleftharpoons}} S_{C_{\mathfrak{b}}}$$

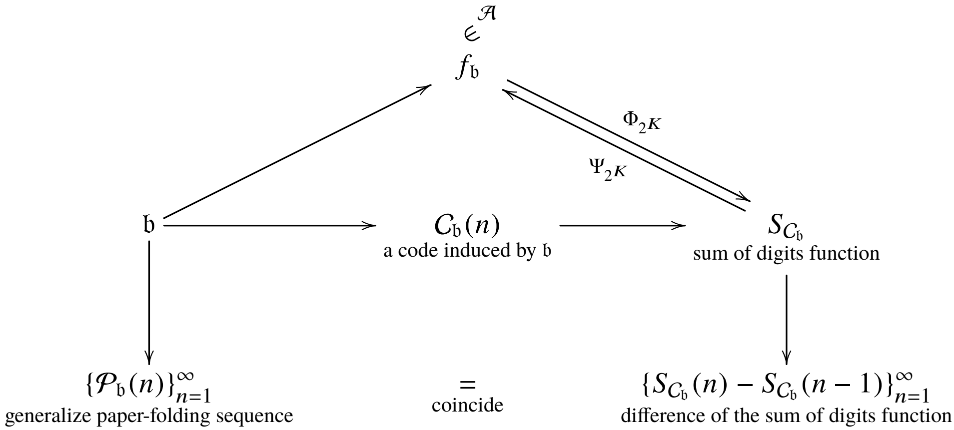
Moreover, we can calculate $f_{\mathfrak{b}}$ as follows:

$$f_{\mathfrak{b}}(n) = \sum_{k=1}^{K} b_k f_0 \left(\frac{n}{2^{k-1}} \right)$$

Where

$$f_0(x) := \begin{cases} \chi_4(x) & \text{if } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The function $f_{\mathfrak{b}}$ is a $2^{K+1}-$periodic function.
The following diagram summarize this last case:



Theorem 9. *If $\mathfrak{b}, K, C_{\mathfrak{b}}(n)$ and $S_{C_{\mathfrak{b}}}(n)$ are as above then:*

$$\frac{1}{N}\sum_{n=0}^{N-1} S_{C_{\mathfrak{b}}}(n) = \frac{1}{2\log 2^K}\sum_{k=1}^{K} b_k \log N + F\left(\frac{\log N}{\log 2^K}\right).$$

Where F is $1-$periodic function, continuous, nowhere differentiable.
Moreover, F admits a Fourier expansion:

$$F(x) = \sum_{k\in\mathbb{Z}} D_k e^{2\pi i k x}$$

With

$$D_k = \begin{cases} \left(\frac{1}{2} - \frac{1}{\log 2^K}\right) L\left(0, f_{\mathfrak{b}}\right) + \frac{L'(0, f_{\mathfrak{b}})}{\log 2^K} & \text{if } k = 0 \\ \\ \dfrac{L\left(\frac{2\pi i k}{\log 2^K}, f_{\mathfrak{b}}\right)}{2\pi i k\left(\frac{2\pi i k}{\log 2^K} + 1\right)} & \text{if } k \neq 0 \end{cases}$$

Where $L(s, f_{\mathfrak{b}})$ is the Dirichlet series with coefficient $f_{\mathfrak{b}}(n)$. This is a generalization of Delange's result in [1].

References

[1] H. Delange *Sur la function sommatoire de la function "somme des chiffres"*, L'Enseignement Math., 21, (1975), 31- 47.

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[3] Y. Kamiya and L. Murata *Certain codes related to generalized paperfolding sequences*, Journal de Theorie des Nombres de Bordeaux, 27 (2015), 149-169.

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