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On the growth of the p-ranks of the Class groups in p-adic analytic Lie extensions

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1 Introduction

Let p be a prime, $\mathbf{K}$ a number field, and let $\mathbf{K}_\infty/\mathbf{K}$ be a $\mathbb{Z}_p$-extension. Iwasawa showed (cfr. [6]) that the size of the μ-invariant is related to the rate of growth of p-ranks of p-class groups in the tower

$$\mathbf{K} \subset \mathbf{K}_1 \subset \mathbf{K}_2 \subset \ldots \subset \mathbf{K}_\infty.$$

He showed in 1958 that the vanishing of the μ-invariant for cyclotomic $\mathbb{Z}_p$-extensions of the rationals is equivalent to certain congruences between Bernoulli numbers and he conjectured that $\mu = 0$ for these extensions. This was verified in 1979 for base fields $\mathbf{K}$ which are abelian over $\mathbb{Q}$ by Ferrero and Washington (cfr. [1]) but it remains an unresolved problem for more general base fields. Iwasawa initially conjectured that his μ-invariant vanishes for all $\mathbb{Z}_p$-extensions, but later, in 1973, he was the first to construct $\mathbb{Z}_p$-extensions with arbitrarily large $\mu-$invariants (cfr. [3]).

As a natural development of this branch of algebraic number theory, Maire's work investigates about the other p-adic Galois groups enjoying the aforesaid property (see [4]).

2 The classical case

Let us start by recalling the main constructions in the simplest case, see [3] or [6]. For $n \geq 1$, let $\mathbf{K}_n = \mathbb{Q}(\zeta_{p^n})$ where ζ_{p^n} is a primitive $p^n - th$ root of unity. As customary we set

$$\mathbf{K}_\infty = \bigcup_{n=1}^{\infty} \mathbf{K}_n.$$

Now, for $p > 2$, Gal $(\mathbf{K}_\infty/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_p^\times$ and $\mathbb{Z}_p^\times$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$ (for $p = 2$, Gal $(\mathbf{K}_\infty/\mathbb{Q})$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$). Therefore the fixed field of $(\mathbb{Z}/p\mathbb{Z})^\times$ as Galois group (over $\mathbb{Q}$) isomorphic to $\mathbb{Z}_p$. We call this field $\mathbb{Q}_\infty$. The extension $\mathbb{Q}_\infty/\mathbb{Q}$ is an archetype for $\mathbb{Z}_p$-extensions, i.e. extensions whose Galois group is isomorphich to $\mathbb{Z}_p$. As shown in [6, Chapter 7] one can find a chain of subfields of $\mathbb{Q}_\infty$:

$$\mathbb{Q} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \bigcup_{n \geq 0} \mathbf{F}_n = \mathbb{Q}_\infty$$

with

$$\text{Gal}\,(\mathbf{F}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}.$$

Iwasawa's theorem can then be stated as follows:

Theorem (Iwasawa). *Let $\mathbb{Q}_\infty$ and $\mathbf{F}_n$ be as above. Let p^{e_n} be the exact power of p dividing the class number of $\mathbf{F}_n$. Then there exist integers $\lambda \geq 0, \mu \geq 0$, and ν, all independent of n, and an integer n_0 such that*

$$e_n = \lambda n + \mu p^n + \nu$$

for all $n \geq n_0$

Sketch of proof. We give a brief outline of the proof following [6, section 13.3]. Let $\Gamma = \mathrm{Gal}\,(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$. Denote by $\mathbf{L}_n$ the maximal unramified abelian p-extension of $\mathbf{F}_n$. Note that $\mathbf{L}_n$ is Galois over $\mathbf{F}_n$ (being maximal). It follows that $X_n \simeq \mathrm{Gal}\,(\mathbf{L}_n/\mathbf{F}_n)$ is isomorphic to the p-Sylow subgroup of the ideal class group of $\mathbf{F}_n$, which we call A_n. Set

$$\mathbf{L} = \bigcup_{n \geq 0} \mathbf{L}_n \qquad \text{and} \qquad X = \mathrm{Gal}\,(\mathbf{L}/\mathbb{Q}_\infty)\,.$$

Note that $\mathbf{L}$ is also Galois extension of $\mathbb{Q}$, and so we set $G = \mathrm{Gal}\,(\mathbf{L}/\mathbb{Q})$. The idea is to made X into a Γ-module and hence a $\Lambda := \mathbb{Z}_p[[T]]$-module. Then one can show that actually X is finitely generated and Λ-torsion. Hence it can be shown that X sits inside an exact sequence of Λ-modules of the form

$$0 \to A \to X \to \left(\bigoplus_{i=1}^{s} \Lambda/(p^{n_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(f_j(T))^{m_j} \right) \to B \to 0,$$

where A and B are finite Λ-modules and each f_j is an irreducible polynomial which is also distinguished (i.e. a monic polynomial whose coefficients, except for the leading coefficient, are all divisible by p). It is not difficult to calculate what happens at the n-th level for modules of the form $\Lambda/(p^n)$ and $\Lambda/(f(T))^m$. One then concludes the proof by transfering back the result to X.

3 Uniform pro-p group

Let Γ be an analytic pro-p group: we can think of Γ as a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$ for a certain integer m. If $p \geq 3$ we say that Γ is powerful if $[\Gamma, \Gamma] \subset \Gamma^p$ ($[\Gamma, \Gamma] \subset \Gamma^4$ in the case $p = 2$). A powerful pro-p group Γ is said uniform if it has no torsion.

The first task is to define $\mathbb{Z}_p[[\Gamma]]$ and this is done by setting $\mathbb{Z}_p[[\Gamma]] := \varprojlim_U \mathbb{Z}_p[[\Gamma/U]]$, where U runs over the open normal subgroups of Γ. Then we set $\Omega := \mathbb{Z}_p[[\Gamma]]/(p)$, and since the rings Ω and $\mathbb{Z}_p[[\Gamma]]$ are local, noetherian and without zero divisor ([2]) each of them has a fractional skew field. Call $Q(\Omega)$ the fractional skew field of Ω. Now if $\mathcal{X}$ is a finitely generated $\mathbb{Z}_p[[\Gamma]]$-module, we define $\mathrm{rk}_\Omega(\mathcal{X})$ to be the $Q(\Omega)$-dimension of $\mathcal{X} \otimes_\Omega Q(\Omega)$.

Finally we set:

$$\mu(\mathcal{X}) = \sum_{i \geq 0} \mathrm{rk}_\Omega \left(\mathcal{X}[p^{i+1}]/\mathcal{X}[p^i] \right),$$

where $\mathcal{X}[p^i]$ is the submodule of the elements of $\mathcal{X}$ killed by p^i.

Let $\mathbf{L}/\mathbf{K}$ be a uniform p-extension: $\mathbf{L}/\mathbf{K}$ is a normal extension whose Galois group $\Gamma := \mathrm{Gal}(\mathbf{L}/\mathbf{K})$ is a uniform pro-p group. We assume furthermore that the set of places of $\mathbf{K}$ that are ramified in $\mathbf{L}/\mathbf{K}$ is finite.

Let $\mathbf{F}/\mathbf{K}$ be a finite subextension of $\mathbf{L}/\mathbf{K}$. We denote by $A(\mathbf{F})$ the p-Sylow subgroup of the class group of $\mathbf{F}$ and put

$$\mathcal{X}_{\mathbf{L}/\mathbf{K}} := \varprojlim_{\mathbf{F}} A(\mathbf{F}),$$

where the limit is taken over all number fields $\mathbf{F}$ in $\mathbf{L}/\mathbf{K}$ with respect the norm map. We have that $\mathcal{X}_{\mathbf{L}/\mathbf{K}}$ is a finitely generated $\mathbb{Z}_p[[\Gamma]]$-module and hence we can associate as above its μ-invariant which is a generalization of the classical μ-invariant introduced by Iwasawa in the particular case $\Gamma = \mathbb{Z}_p$. Set $\mu_{\mathbf{L}/\mathbf{K}} := \mu(\mathcal{X}_{\mathbf{L}/\mathbf{K}})$. A first interesting result about μ for this module was proven by Perbet in [5]:

Theorem. *For $n \gg 0$ one has:*

$$\log |A(\mathbf{K}_n)/p^n| = \mu_{\mathbf{L}/\mathbf{K}} p^{dn} \log p + O(np^{d(n-1)}),$$

where d is the dimension of Γ as an analytic variety.

We will say that a number field $\mathbf{K}$ is called p-rational if the Galois group of the maximal pro-p-extension of $\mathbf{K}$ unramified outside p is pro-p free. The crucial property of p-rational fields is, informally, that in terms of certain maximal p-extensions with restricted ramification, they behave especially well, almost as well as $\mathbb{Q}$.

The main result can be summarized as follows:

Theorem (Hajir-Maire). *Let* Γ *be a uniform pro-*p *group having an automorphism* τ *of order* m *with fixed-point-free action, where* $m \geq 3$ *is co-prime to* p.
Assume $\mathbf{F}_0$ *is a totally imaginary number field admitting a cyclic extension* $\mathbf{F}/\mathbf{F}_0$ *of degree* m *such that* $\mathbf{F}$ *is* p*-rational.*
For any given integer μ_0*, there exists a cyclic degree* p *extension* $\mathbf{K}'$ *over* $\mathbf{K}(\zeta_p)$ *and a* Γ*-extension* $\mathbf{L}'/\mathbf{K}'$ *of* $\mathbf{K}'$ *whose* μ*-invariant verifies:*

$$\mu_{\mathbf{L}'/\mathbf{K}'} \geq \mu_0.$$

If p *is a regular prime and* m *is an odd divisor of* $p-1$ *we can choose* $\mathbf{F} = \mathbb{Q}(\zeta_{p^n})$ *for any* $n \geq 1$.

References

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