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Prime numbers in short intervals: the Selberg integral and its generalisations

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The central problem of Analytic Number Theory is the distribution of prime numbers. The answers to the questions that naturally arise from this problem are only partially known, even assuming powerful and, as yet, unproved hypotheses like Riemann's. Here we are interested in the distribution of prime numbers in "short intervals". With this term we mean intervals of the form $(x, x + y]$, where $y = o(x)$.

Recall the prime-counting function

$$\pi(x) \stackrel{\text{def}}{=} \#\{p \le x \mid p \text{ is prime}\} \sim \mathrm{li}(x) \stackrel{\text{def}}{=} \int_2^x \frac{dt}{\log t}$$

and the Chebyshev function

$$\psi(x) \stackrel{\text{def}}{=} \sum_{n \le x} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function defined to be equal to $\log(p)$ if $n = p^\alpha$ for some p prime and positive integer α, and zero otherwise. It is well known that the Riemann Hypothesis (RH, for short) is equivalent to either of the two statements

$$\pi(x) = \mathrm{li}(x) + O\left(x^{\frac{1}{2}} \log(x)\right) \quad \text{or} \quad \psi(x) = x + O\left(x^{\frac{1}{2}} (\log x)^2\right).$$

Looking to the “additive" form of the expected main term of both π and ψ, a natural question arises.

Question 1. *For $y \le x$, is it true that*

$$\pi(x+y) - \pi(x) \sim \int_x^{x+y} \frac{dt}{\log t} \quad \textit{or} \quad \psi(x+y) - \psi(x) \sim y \quad ? \quad (1)$$

In some applications it is sufficient to know that such asymptotic relations hold for most values of y. For measuring precisely what “usually" means, Selberg introduced the variance of the primes in short intervals

$$J(x,\theta) \stackrel{\text{def}}{=} \int_x^{2x} |\psi(t+\theta t) - \psi(t) - \theta t|^2 dt, \quad (2)$$

where $\theta \in [0,1]$ is essentially y/x. On the Riemann Hypothesis we have that

$$J(x,\theta) \ll x^2\theta(\log(2/\theta)^2) \quad \text{uniformly for } x^{-1} \le \theta \le x.$$

It means that in this range of values for θ,

$$\psi(t+\theta t) - \psi(t) = \theta t + O\left((\theta x)^{\frac{1}{2}} \log x\right) \quad \text{for “almost all" } t \in [x, 2x].$$

We now assume RH until the end. As is customary, we denote the generic non-trivial zero of the Riemann ζ-function by $\rho = \frac{1}{2} + i\gamma$. Consider the Montgomery pair-correlation function

$$F(x,T) \stackrel{\text{def}}{=} \sum_{\gamma_1,\gamma_2 \in [0,T]} \frac{4x^{i(\gamma_1-\gamma_2)}}{4 + (\gamma_1-\gamma_2)^2}.$$

In [5] Montgomery proved that $F(x,T) \sim \frac{T}{2\pi}\log x$ as $T \to +\infty$ uniformly for $T^{\varepsilon} \le x \le T$ and conjectured that $F(x,T) \sim \frac{T}{2\pi}\log T$ as $T \to +\infty$ uniformly for $T \le x \le T^{A}$. It means that only the “diagonal" terms (where $\gamma_1 = \gamma_2$) of the sum give a contribution. We are interested in the connection between hypothetical asymptotic formulae for J and F. More precisely, if we write J and F in expansions like

$$J(x,\theta) = \frac{3}{2}x^2\theta(\log(1/\theta) + 1 - \gamma - \log(2\pi)) + R_J(x,\theta)$$

and

$$F(x,T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + R_F(x,T),$$

then we want to answer to the following

Question 2. *Is it possible to compare the size of the error terms $R_J(x,\theta)$ and $R_F(x,T)$ in suitable ranges of uniformity?*

Assuming RH, Montgomery and Soundararajan [6] proved that $R_J(x,\theta) = o(x^2\theta)$ if and only if $R_F(x,T) = o(T)$. In [2], Languasco, Perelli and Zaccagnini studied relations between hypothetical bounds of the type

$$R_J(x,\theta) = O(x^2\theta^{1+\alpha}) \quad \text{and} \quad R_F(x,T) = O(T^{1-\beta}).$$

Their results are as general as they are cumbersome, so, for simplicity, we state a weakened and simplified version of such results, leaving out log-powers and uniformity in the various parameters. Essentially, for $\alpha, \beta > 0$, we have

$$R_J(x,\theta) \ll x^2\theta^{1+\alpha} \qquad \Longrightarrow \qquad R_F(x,T) \ll T^{1-\frac{\alpha}{\alpha+3}},$$
$$R_F(x,T) \ll T^{1-\beta} \qquad \Longrightarrow \qquad R_J(x,\theta) \ll x^2\theta^{1+\frac{\beta}{2}}.$$

We now introduce a new pair-correlation function and connect it to a more general form of the Selberg integral. Let $\tau \in [0,1]$ and define

$$F(x,T,\tau) \stackrel{\text{def}}{=} \sum_{\gamma_1,\gamma_2\in[-T,T]} \frac{4x^{i(\gamma_1-\gamma_2)}}{4+\tau^2(\gamma_1-\gamma_2)^2}.$$

Of course $F(x,T,1)$ is essentially the same as $F(x,T)$. Moreover $F(x^{1/\tau},T,\tau)$ is the pair-correlation function for $Z_\tau(s)=\zeta(s/\tau)$, where Z_τ is (almost) an element of the Selberg Class of degree $\frac{1}{\tau}$ and conductor $\left(\frac{1}{\tau}\right)^{1/\tau}$. Continuing with the properties of $F(x,T,\tau)$, we note that $F(x,T,0)=|\Sigma(x,T)^2|$, where

$$\Sigma(x,T) \stackrel{\text{def}}{=} \sum_{|\gamma|\le T} x^{i\gamma}$$

is the exponential sum that appears in Landau's explicit formula for $\psi(x)$. We also remark that $F(x,T,\tau)$ is difficult to estimate for τ very small (say, $\tau \le 1/T$) because in this case the trivial bound $F(x,T,\tau) \ll \min(T,\tau^{-1})T\log^2 T$ becomes very large.

From now on we assume $\tau > 0$ in order to avoid trivial statements. Let

$$J(x,\tau,\theta) \stackrel{\text{def}}{=} \int_x^{x(1+\tau)} |\psi(t+\theta t)-\psi(\theta)-\theta t|^2 dt$$

Here we are dealing with "short intervals" in two different ways. The obvious conjecture is $J(x,\tau,\theta) \ll x^{2+\varepsilon}\tau\theta$.

There is a conjecture of Gonek involving the behaviour of $\Sigma(x,T)$. It states that $\Sigma(x,T) \ll Tx^{-1/2+\varepsilon} + T^{1/2}x^{\varepsilon}$ for $x,T \ge 2$. This conjecture and an obvious generalisation of Montgomery's "justify us" to work by assuming the following

Assumption (Hypothesis $H(\eta)$)**.** *For some fixed $\eta > 0$ and every $\varepsilon > 0$ we have*

$$F(x,T,\tau) \ll Tx^{\varepsilon} \quad \textit{uniformly for} \quad \begin{cases} x^\eta \le T \le x \\ x^\eta/T \le \tau \le 1. \end{cases}$$

By using such assuption, in [3], Languasco, Perelli and Zaccagnini proved the following result.

Theorem 1. *If assumption $H(\eta)$ holds for some $\eta \in (0,1)$, then*

$$J(x,\tau,\theta) \ll x^{2+\varepsilon}\tau\theta$$

uniformly for $x^{-1} \le \theta \le x^{-\eta}$ and $\theta x^{\eta} \le \tau \le 1$. Moreover if assumption $H(\eta)$ holds for some $\eta \in (0, 1/2-5\varepsilon)$ (for $\varepsilon > 0$ small), then

$$\psi(x+y)-\psi(x) = y + \begin{cases} O(y^{2/3}x^{\eta/3+\varepsilon}) & \text{for } x^{\eta+5\varepsilon} \le y \le x^{1/2} \\ O(y^{1/3}x^{1/6+\eta/3+\varepsilon}) & \text{for } x^{1/2} \le y \le x^{1-\eta}. \end{cases}$$

As an immediate consequence we have

$$\psi(x+y)-\psi(x) = y + O(y^{1/2}x^{\varepsilon})$$

for “almost all" $x \in [x, x(1+\tau)]$ and $y \in [1, x^{1-\eta}]$.

In [4] Languasco, Perelli and Zaccagnini gave an asymptotic result for $F(x,T,\tau)$. Let

$$S(x,\tau) \stackrel{\text{def}}{=} \sum_{n\ge 1} \frac{\Lambda^2(n)}{n} a^2(n,x,\tau)$$

where

$$a(n,x,\tau) \stackrel{\text{def}}{=} \begin{cases} (n/x)^{1/\tau} & \text{if } n \le x \\ (x/n)^{1/\tau} & \text{if } n > x. \end{cases}$$

Then the following Theorem holds.

Theorem 2. *As $x \to +\infty$ we have*

$$F(x,T,\tau) \sim \frac{T}{\pi}\frac{S(x,\tau)}{\tau} + \frac{T\log^2 T}{\pi\tau x^{2/\tau}} + \textit{smaller order terms}$$

uniformly for $\tau \ge 1/T$, provided that $TS(x,\tau) = \infty(\max(x, (\log T)^3/\tau))$.

Of course, this reduces to Montgomery’s result for $\tau = 1$. We remark that the Theorem shows the same phenomenon of yielding an asymptotic formula only at “extreme ranges". Notice that if τ is not too small, say $\tau \geq x^{\varepsilon-1}$, by the Brun-Titchmarsh inequality it follows that $S(x,\tau) \ll \tau \log x$. Moreover, if $y \leq x$ and

$$\psi(x+y) - \psi(x) \sim y \quad \text{uniformly for } y \geq x^{\beta+\varepsilon}$$

then

$$S(x,\tau) \sim \tau \log x \quad \text{uniformly for } \tau \geq x^{\beta+\varepsilon-1}.$$

However, S is erratic for $\tau \leq 1/x$. Essentially, it reduces to the single term given by the prime power closest to x.

Finally, in [4], Languasco, Perelli and Zaccagnini also gave the following asymptotic result.

Theorem 3. *Assume the "Generalized Montgomery Conjecture". Then*

$$J(x,\tau,\theta) \sim \left(1 + \frac{\tau}{2}\right) \tau\theta \log(1/\theta)$$

uniformly for $1/x \leq \theta \leq x^{-\varepsilon}$ *and* $\theta^{1/2-\varepsilon} \leq \tau \leq 1$.

Of course, the first factor here is relevant only if $\tau \gg 1$, when Theorem 3 is a consequence of earlier results. The proof requires a suitable, stronger version of the technique introduced by Goldson and Montgomery in [1], with particular care for the τ-uniformity aspect.

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