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Kloosterman sums and shifted character sums with multiplicative coefficients

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1 Introduction on Kloosterman sum

Given a real number t, we write $e(t) = e^{2\pi i t}$. The sum

$$S(a,\, b;\, q) = \sum_{\substack{x=1 \\ (x,\, q)=1}}^{q} e\Big(\frac{ax + b\bar{x}}{q}\Big)$$

where $\bar{x}$ satisfies $\bar{x}x \equiv 1 \pmod q$, and is unique modulo q, is called a Kloosterman sum. Kloosterman sum play an important role in number theory. There is systematic and deep study on this type of sums. Kloosterman introduced this type of sums as early as 1926. His purpose was to study positive integer solutions of the quadratic diagonal form

$$N = a_1 n_1^2 + a_2 n_2^2 + a_3 n_3^2 + a_4 n_4^2,$$

where a_i are fixed positive integers. This problem is a generalization of Lagrange four squares theorem, the purpose of which is to determine for which coefficients $(a_1,\, a_2,\, a_3,\, a_4)$, all sufficiently large N can be expressed in this form. Now it is difficult to apply Hardy-Littlewood circle method directly, and Kloosterman had to make an improvement on the circle method. During the proof he met the sum

$$\sum_{\substack{x=1\\(x,\,q)=1}}^{X} e\Big(\frac{b\bar{x}}{q}\Big),$$

which is called an incomplete Kloosterman sum.

If $(b,\ q) = 1,\ X = q$, the above sum is a Ramanujan sum

$$\sum_{\substack{x=1\\(x,\,q)=1}}^{q} e\Big(\frac{b\bar{x}}{q}\Big) = \sum_{\substack{y=1\\(y,\,q)=1}}^{q} e\Big(\frac{y}{q}\Big) = \mu(q).$$

The incomplete sum can be transformed to a complete sum by a standard technique. Therefore

$$\sum_{\substack{x=1\\(x,\,q)=1}}^{X} e\Big(\frac{b\bar{x}}{q}\Big)\Big| \le (1+\log q) \max_{1\le a\le q} |S(a,\ b;\ q)|.$$

Now the problem is changed into finding an estimate for $S(a,\ b;\ q)$.

This sum can be traced back to a work of H. Poincaré in 1911. It is named after Kloosterman since he gave a non-trivial estimate for this sum for the first time. From now on we restrict to prime number $p = q$ in the treatment of Kloosterman sums. Firstly he considered the mean value

$$\sum_{r=0}^{p-1}\sum_{s=0}^{p-1} |S(r,\ s;\ p)|^4.$$

This mean value has an arithmetic meaning, i.e. is the number of solutions of the equation system

$$x_1 + x_2 - x_3 - x_4 \equiv 0, \qquad \bar{x}_1 + \bar{x}_2 - \bar{x}_3 - \bar{x}_4 \equiv 0 \pmod p.$$

By an elementary discussion, the number of solutions does not exceed $3p^3(p-1)$.

Then for the fixed $a,\ b,\ p \nmid b$, there are at least $p-1$ terms $|S(a,\ b;\ p)|^4$ in the above mean value. Combining all of the above, we get

$$|S(a,\ b;\ p)| < 3^{\frac{1}{4}} p^{\frac{3}{4}}, \qquad p \nmid b.$$

This shows that in the above incomplete Kloosterman sum, if $X > p^{\frac{3}{4}+\varepsilon}$, there is some cancellation. By this estimate, Kloosterman can solve the problem on the quadratic diagonal form in four variables.

In 1948, Weil proved a stronger result

$$|S(a,\ b;\ p)| \leq 2p^{\frac{1}{2}}, \qquad p \nmid b,$$

which is a corollary of his proof of Riemann Hypothesis for curves in finite fields. This estimate is almost best possible, and is used in many applications of Kloosterman sum.

2 Applications

1) It is conjectured that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be expressed as

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2,$$

where p_i is prime number. This is a development of Lagrange four squares theorem. The case for five primes was solved by L. Hua. The case for four primes is very difficult. In 1994, Brüdern and Fouvry proved that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be expressed as

$$N = P_1^2 + P_2^2 + P_3^2 + P_4^2,$$

where the number of prime factors of every P_i is at most 34. In the proof, the improvement on circle method by Kloosterman and the estimate for Kloosterman sum were used. Yingchun Cai improved the number 34 to 13. Recently Lilu Zhao improved it to 5.

2) Let p be prime, $(a, p) = 1$. Solve

$$mn \equiv a \pmod p,$$

where positive integers m and n are small as possible.

Write $M(a)$ as the minimum of $\max(m, n)$. It is obvious that

$$M(p-1) \geq \sqrt{p-1}, \qquad M(a) \leq p-1.$$

Is there a better upper bound for $M(a)$?

Write

$$A_n = \begin{cases} 1, & \text{if } mn \equiv a \pmod p \text{ has solution } 1 \leq m \leq M, \\ 0, & \text{otherwise.} \end{cases}$$

By a standard technique, we get

$$\begin{aligned} \Big| \sum_{0<n\leq M} A_n - \frac{M^2}{p} \Big| &\leq \log p \max_{1\leq k<p} \Big| \sum_{n=1}^{p} \sum_{\substack{m=1 \\ mn\equiv a \pmod p}}^{M} e\Big(\frac{kn}{p}\Big) \Big| \\ &= \log p \max_{1\leq k<p} \Big| \sum_{m=1}^{M} e\Big(\frac{ka\bar{m}}{p}\Big) \Big| \\ &< 4(\log p)^2 p^{\frac{1}{2}}. \end{aligned}$$

The sum

$$\sum_{0<n\leq M} A_n$$

is the number of solutions of the equation

$$mn \equiv a \pmod p, \qquad 1 \leq m, n \leq M.$$

Hence, if

$$\frac{M^2}{p} \geq 4(\log p)^2 p^{\frac{1}{2}},$$

then the above equation must has a solution, which means

$$M(a) \leq 2(\log p)p^{\frac{3}{4}}.$$

The improvement on the exponent $\frac{3}{4}$ is an open problem.

3) Let $d(n)$ be divisor function. We consider the behaviour of the sum

$$\sum_{n\leq x} d(n)d(n+1).$$

This sum counts the number of integer arrays (a, b, r, s) which satisfy

$$ab \leq x, \qquad ab+1 = rs.$$

We can get the condition

$$ab \equiv -1 \pmod r.$$

For the fixed r, we calculate how many a, b. A similar problem with the above one appears, which is on $M(-1)$. By the similar discussion, we get an asymptotic formula

$$\sum_{n\leq x} d(n)d(n+1) = xQ(\log x) + O(x^{\frac{5}{6}+\varepsilon}),$$

where ε a sufficient small positive constant, Q is some quadratic polynomial. For any positive integer a, the sum $\sum_{n\leq x} d(n)d(n+a)$ can be dealt with in the same way. These sums are interested since they appear in the Riemann zeta function theory.

We see the integral

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 dt.$$

Write $|\zeta|^4 = \zeta^2\overline{\zeta^2}$. Expand this product to produce a double sum the near diagonal terms contain $d(n)d(n+a)$. Then the problem is changed into that for $\sum_{n\leq x} d(n)d(n+a)$. Heath-Brown proved that

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 dt = TF(\log T) + O(T^{\frac{7}{8}+\varepsilon}),$$

where F is some fourth power polynomial. Motohashi improved the exponent $\frac{7}{8}$ to $\frac{2}{3}$. He used the mean value theory of Kloosterman sums which was developed by Kuznetsov and Iwaniec.

4) Iwaniec made further development on the classic fourth power mean value of ζ function. His object is to estimate the sixth power of ζ function. He proved that

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 \Big|\sum_{n\le N} \frac{a(n)}{n^{\frac{1}{2}+it}}\Big|^2 dt \ll T^{1+\varepsilon},$$

where $a(n) = O(1)$, $N \ll T^{\frac{1}{10}}$. He used the estimate for the Kloosterman sum.

Afterwards, Iwaniec and Deshouillers, Watt improved the exponent $\frac{1}{10}$ into $\frac{1}{5}$ and $\frac{1}{4}$. They used the estimate for the Kloosterman sum.

In 2014, Chaohua Jia and A. Sankaranarayanan proved that

$$\sum_{n\le x} d^2(n) = xP(\log x) + O(x^{\frac{1}{2}}(\log x)^5),$$

where $P(x)$ is some cubic polynomial. We made some refinement on the work of Iwaniec and also used the estimate for the Kloosterman sum.

3 Further development

Let us see the Kloosterman sum with some coefficients. In 1988, D. Hajela, A. Pollington, B. Smith proved that if $(b, q) = 1$, then

$$\sum_{\substack{n\le N \\ (n,q)=1}} \mu(n)e\Big(\frac{b\bar{n}}{q}\Big) \ll Nq^{\varepsilon}\Big(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{3}{10}}(\log N)^{\frac{11}{5}}}{N^{\frac{1}{5}}}\Big).$$

This estimate is non-trivial for $(\log N)^{5+10\varepsilon} \ll q \ll N^{\frac{2}{3}-3\varepsilon}$.

Afterwards, P. Deng, G. Wang and Z. Zeng independently proved that

$$\sum_{\substack{n\le N\\(n,q)=1}} \mu(n)e\Big(\frac{b\bar{n}}{q}\Big) \ll Nq^{\varepsilon}\Big(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{1}{5}}(\log N)^{\frac{13}{5}}}{N^{\frac{1}{5}}}\Big).$$

This estimate is non-trivial for $(\log N)^{5+\varepsilon} \ll q \ll N^{1-\varepsilon}$.

P. Deng pointed out that under GRH, one can get

$$\sum_{\substack{n\le N\\(n,q)=1}} \mu(n)e\Big(\frac{b\bar{n}}{q}\Big) \ll q^{\frac{1}{2}}N^{\frac{1}{2}+\varepsilon},$$

which is non-trivial for $q \ll N^{1-4\varepsilon}$. Therefore to break through the limitation $q \le N$ is the next direction of development.

In 1998, Fouvry and Michel proved that if q is a prime number, $P(x)$ and $Q(x)$ are coprime monic polynomials on $\mathbb{Z}[x]$, $g(x) = \frac{P(x)}{Q(x)}$ is a rational function, then for $N \le q$, one has

$$\sum_{\substack{p\le N\\(Q(p),q)=1}} e\Big(\frac{g(p)}{q}\Big) \ll q^{\frac{3}{16}+\varepsilon}N^{\frac{25}{32}},$$

where p is the prime number. This estimate is non-trivial for $N \le q \ll N^{\frac{7}{6}-7\varepsilon}$.

They also proved that for $N \le q$,

$$\sum_{\substack{n\le N\\(Q(n),q)=1}} \mu(n)e\Big(\frac{g(n)}{q}\Big) \ll q^{\frac{3}{16}+\varepsilon}N^{\frac{25}{32}}.$$

In 2011, Fouvry and Shparlinski proved that for $(b, q) = 1$, $N^{\frac{3}{4}} \le q \le N^{\frac{4}{3}}$, one has

$$\sum_{\substack{N<p\le 2N\\(p,q)=1}} e\Big(\frac{b\bar{p}}{q}\Big) \ll q^{\varepsilon}(q^{\frac{1}{4}}N^{\frac{2}{3}} + N^{\frac{15}{16}}).$$

This estimate is non-trivial for $N^{\frac{3}{4}} \le q \ll N^{\frac{4}{3}-6\varepsilon}$.

They also proved that

$$\sum_{\substack{N<p\le 2N\\(p,q)=1}} e\Big(\frac{b\bar{p}}{q}\Big) \ll Nq^{\varepsilon}\Big(\frac{(\log N)^2}{q^{\frac{1}{2}}}+\frac{q^{\frac{1}{4}}(\log N)^{\frac{3}{2}}}{N^{\frac{1}{5}}}\Big).$$

This estimate is non-trivial for $(\log N)^{6+\varepsilon} \ll q \ll N^{\frac{4}{5}-\varepsilon}$.

There are corresponding results for $\mu(n)$.

Naturally one would consider more general situation. When Ke Gong visited Montreal University, Professor Granville suggested him to study the non-trivial estimate for

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)e\Big(\frac{b\bar{n}}{q}\Big)$$

where $f(n)$ is a multiplicative function satisfying $|f(n)| \le 1$.

In the above results, one can apply Vaughan's method in which properties that

$$\sum_{d\,|\,n}\Lambda(d) = \log n$$

or

$$\sum_{d\,|\,n}\mu(d) = \begin{cases} 1, & \text{if } n = 1,\\ 0, & \text{if } n > 1.\end{cases}$$

are used.

But in our situation, we know nothing on $\sum_{d\,|\,n} f(d)$ so that Vaughan's method does not work. We have to seek the new method. Fortunately, we find that the finite version of Vinogradov's inequality which is used by Bourgain, Sarnak and Ziegler is available.

Recently, Ke Gong and Chaohua Jia proved that if $f(n)$ is a multiplicative function, $|f(n)| \le 1$, $q \le N^2$, $(b, q) = 1$, then

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)e\Big(\frac{b\bar{n}}{q}\Big) \ll \sqrt{\frac{d(q)}{q}}N(\log\log 6N)+q^{\frac{1}{4}+\frac{\varepsilon}{2}}N^{\frac{1}{2}}(\log 6N)^{\frac{1}{2}}+\frac{N}{\sqrt{\log\log 6N}}.$$

This estimate is non-trivial for

$$(\log\log 6N)^{2+\varepsilon} \ll q \ll N^{2-5\varepsilon}.$$

If $f(n) = \mu(n)$, we can get a bigger range of non-trivial estimate than before. But for the sum on prime numbers, our method is not available.

4 Shifted character sum

We have corresponding result on the shifted character sum.

Let q be a prime number, $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q.

Since the 1930s, I. M. Vinogradov had begun the study on character sums over shifted primes

$$\sum_{p \le N} \chi(p + a),$$

and obtained deep results, where p is prime number. His best known result is a nontrivial estimate for the range $N^{\varepsilon} \le q \le N^{\frac{4}{3}-\varepsilon}$, where ε is a sufficiently small positive constant, which lies deeper than the direct consequence of GRH.

Later, Karatsuba widen the range to $N^{\varepsilon} \le q \le N^{2-\varepsilon}$, where Burgess's method was applied.

For the Möbius function $\mu(n)$, one can get same results on sums

$$\sum_{n \le N} \mu(n)\chi(n + a)$$

as that on sums over shifted primes.

Recently, Ke Gong and Chaohua Jia used the finite version of Vinogradov's inequality to prove the following result:

If $f(n)$ is a multiplicative function satisfying $|f(n)| \le 1$, q $(\le N^2)$ is a prime number and $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q, then we have

$$\sum_{n\le N} f(n)\chi(n+a) \ll \frac{N}{q^{\frac{1}{4}}}\log\log(6N)+q^{\frac{1}{4}}N^{\frac{1}{2}}\log(6N)+\frac{N}{\sqrt{\log\log(6N)}}.$$

This estimate is non-trivial for

$$(\log\log(6N))^{4+\varepsilon} \ll q \ll \frac{N^2}{(\log(6N))^{4+\varepsilon}}.$$

References

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