Lecture 5

January 15, 2018

1 Divisors

For any positive integer n let $d(n)$ denote the number of positive integers which divide n.

Theorem 1.

$$\sum_{m=1}^{n} d(m) = \sum_{m=1}^{n} \left\lfloor \frac{n}{m} \right\rfloor = n \log n + (2\gamma - 1)n + O(n^{1/2}).$$

Proof. Let D_n be the region in the upper right hand quadrant not containing the x or the y axis and under (and including) the hyperbola $xy = n$. That is,

$$D_n = \{(x, y) \in \mathbb{R}^2 \;:\; x > 0,\ y > 0,\ xy \le n\}.$$

Every point with integer coefficients in D_n is contained in a hyperbola $xy = m$ for some positive integer $m \le n$. Thus, $\sum_{m=1}^{n} d(m)$ is the number of points with integer coordinates in D_n. Note that if $x = m$ is fixed, then the number of points in D_n having this value of x is $\left\lfloor \frac{n}{m} \right\rfloor$. Hence, the number of points in D_n is also $\sum_{m=1}^{n} \left\lfloor \frac{n}{m} \right\rfloor$.

In order to estimate how many points are in D_n, we first remark that the number of points above the line $y = x$ is the same as the number of points below. Thus,

$$\begin{aligned}
\sum_{m=1}^{n} \left\lfloor \frac{n}{m} \right\rfloor &= 2 \sum_{x=1}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{x} \right\rfloor - \lfloor x \rfloor \right) + \lfloor \sqrt{n} \rfloor \\
&= 2 \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{k} - k + O(1) \right) + \lfloor \sqrt{n} \rfloor \\
&= \left(2n \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \frac{1}{k} \right) - 2 \left(\frac{\lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor + 1)}{2} \right) + O\left(\sqrt{n}\right). \qquad (1)
\end{aligned}$$

By Theorem 1 of Lecture 2,

$$\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} = \log\lfloor\sqrt{n}\rfloor + \gamma + O\left(\frac{1}{\sqrt{n}}\right). \tag{2}$$

Inserting estimate (2) into (1) we get

$$\sum_{m=1}^{n}\left\lfloor\frac{n}{m}\right\rfloor = 2n\left(\log\lfloor\sqrt{n}\rfloor + \gamma + O\left(\frac{1}{\sqrt{n}}\right)\right) - (n + O(\sqrt{n})) + O(\sqrt{n}). \tag{3}$$

Since

$$\lfloor\sqrt{n}\rfloor = \sqrt{n} - \{\sqrt{n}\} = \sqrt{n} + O(1),$$

we have

$$\begin{aligned}
\log(\lfloor\sqrt{n}\rfloor) &= \log(\sqrt{n} + O(1)) \\
&= \log\left(\sqrt{n}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\right) \\
&= \log(\sqrt{n}) + \log\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \log(\sqrt{n}) + O\left(\frac{1}{\sqrt{n}}\right). \qquad (4)
\end{aligned}$$

Thus, using estimate (4) into estimate (3), we get

$$\sum_{m=1}^{n}\left\lfloor\frac{n}{m}\right\rfloor = n\log n + (2\gamma - 1)n + O(\sqrt{n}),$$

which is what we wanted to prove. □

2 The Prime Number Theorem

We are now ready to prove the Prime Number Theorem.

Theorem 2.

$$\pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

Proof. By Problem 1 of Lecture 2 (which you should have done), it suffices to prove that $\psi(x) \sim x$. Put

$$F(x) = \sum_{n \le x} \left(\psi\left(\frac{x}{n}\right) - \left\lfloor \frac{x}{n} \right\rfloor + 2\gamma \right).$$

By Möbius inversion formula Theorem 1 of Lecture 4, we have

$$\psi(x) - \lfloor x \rfloor + 2\gamma = \sum_{n \le x} \mu(n) F\left(\frac{x}{n}\right).$$

It remains to show that

$$\sum_{n \le x} \mu(n) F\left(\frac{x}{n}\right) = o(x) \qquad (x \to \infty). \tag{5}$$

To do this, we first estimate $F(x)$. We have

$$F(x) = \sum_{n \le x} \psi\left(\frac{x}{n}\right) - \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor + 2\gamma \lfloor x \rfloor.$$

Also

$$\begin{aligned}
\sum_{n \le x} \psi\left(\frac{x}{n}\right) &= \sum_{n \le x} \sum_{m \le x} \Lambda(m) = \sum_{n \le x} \Lambda(n) \left(\sum_{m \le x/n} 1 \right) \\
&= \sum_{n \le x} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{p^k \le x} (\log p) \left\lfloor \frac{x}{p^k} \right\rfloor \\
&= \sum_{p \le x} \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots \right) \log p \\
&= \log(\lfloor x \rfloor !) = \sum_{n \le x} \log n,
\end{aligned}$$

and, as in the proof of Theorem 2 of Lecture 2,

$$\sum_{n \le x} \log n = x \log x - x + O(\log x);$$

hence,

$$\sum_{n \le x} \psi\left(\frac{x}{n}\right) = x \log x - x + O(\log x).$$

Further, by Theorem 1 in today's lecture, we have

$$
\begin{aligned}
\sum_{n=1}^{x} \left\lfloor \frac{x}{n} \right\rfloor &= \sum_{n=1}^{\lfloor x \rfloor} \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor \\
&= \lfloor x \rfloor \log \lfloor x \rfloor + (2\gamma - 1)\lfloor x \rfloor + O(x^{1/2}) \\
&= (x - \{x\}) \log(x - \{x\}) + (2\gamma - 1)(x - \{x\}) + O(x^{1/2}) \\
&= x \log x + x \log \left(1 - \frac{\{x\}}{x}\right) + (2\gamma - 1)x + O(x^{1/2}) \\
&= x \log x + (2\gamma - 1)x + O(x^{1/2}),
\end{aligned}
$$

where we used the fact that

$$
\log \left(1 - \frac{\{x\}}{x}\right) = \log \left(1 + O\left(\frac{1}{x}\right)\right) = O\left(\frac{1}{x}\right).
$$

Thus,

$$
\sum_{n=1}^{\lfloor x \rfloor} \left\lfloor \frac{x}{n} \right\rfloor = x \log x + (2\gamma - 1)x + O(x^{1/2}).
$$

Therefore,

$$
\begin{aligned}
F(x) &= (x \log x - x + O(x^{1/2})) - (x \log x + (2\gamma - 1)x + O(x^{1/2})) + (2\gamma x + O(1)) \\
&= O(x^{1/2}).
\end{aligned}
$$

Thus, there is a positive constant c such that

$$
|F(x)| \le c x^{1/2} \qquad \text{for all } x \ge 1.
$$

Now let t be an integer larger than 1. Then

$$
\begin{aligned}
\left| \sum_{n < x/t} \mu(n) F\left(\frac{x}{n}\right) \right| &\le \sum_{n \le x/t} \left| F\left(\frac{x}{n}\right) \right| \le c \sum_{n \le x/t} \left(\frac{x}{n}\right)^{1/2} \\
&\le c x^{1/2} \left(1 + \int_1^{x/2} \frac{du}{u^{1/2}}\right) \le c x^{1/2} \left(1 + 2u^{1/2} \Big|_{u=1}^{u=t/2}\right) \\
&\le c x^{1/2} \left(1 + 2\left(\frac{x}{t}\right)^{1/2} - 2\right) \le \frac{2cx}{t^{1/2}}. \qquad (6)
\end{aligned}
$$

Observe that F is a step function. In particular, if a is an integer and $a \le x < a+1$, then $F(x) = F(a)$. Therefore,

$$\begin{aligned}\sum_{x/t<n\le x} \mu(n) F\left(\frac{x}{n}\right) &= F(1) \sum_{x/2<n\le x} \mu(n) + F(2) \sum_{x/3<n\le x/2} \mu(n) + \\ &\cdots +F(t-1) \sum_{x/t<n\le x/(t-1)} \mu(n).\end{aligned}$$

Thus,

$$\begin{aligned}\left|\sum_{x/t<n\le x} \mu(n) F\left(\frac{x}{n}\right)\right| &\le |F(1)| \left|\sum_{x/2<n\le x} \mu(n)\right| + \cdots + |F(t-1)| \left|\sum_{x/t<n\le x/(t-1)} \mu(n)\right| \\ &\le (|F(1)| + \cdots + |F(t-1)|) \max_{2\le i\le t} \left|\sum_{x/i<n\le x/(i-1)} \mu(n)\right|. \qquad (7)\end{aligned}$$

But

$$\sum_{x/i<n\le x/(i-1)} \mu(n) = \sum_{n\le x/(i-1)} \mu(n) - \sum_{n\le x/i} \mu(n) = o(x) \qquad \text{as } x\to\infty$$

x/i tends to infinity. Now we have all we need. Let $\varepsilon \in (0,1)$. Choose $t = \lfloor 1/\varepsilon^{1/3} \rfloor$. There exists x_ε such that

$$\left|\sum_{n\le y} \mu(n)\right| \le \varepsilon x,$$

for all $y > x_\varepsilon$. Thus, if x is such that $x > t y_\varepsilon$, then $x/i > y_\varepsilon$ for all $i = 1, \ldots, t$ and so

$$\left|\sum_{x/i<n\le x/(i-1)} \mu(n)\right| \le \left|\sum_{n\le x/(i-1)} \mu(n)\right| + \left|\sum_{n\le x/i} \mu(n)\right| < \varepsilon \left(\frac{x}{i} + \frac{x}{i-1}\right) \le 2\varepsilon x$$

for all $i = t, t-1, \ldots, 2$. Now inequalities (7) and (3) give

$$\left|\sum_{x/t<n\le x} \mu(n) F\left(\frac{x}{n}\right)\right| \le 2\varepsilon x \sum_{i=1}^{t-1} |F(i)| \le 2\varepsilon x \sum_{i=1}^{t-1} c i^{1/2} \le 2c\varepsilon t^{3/2} x,$$

which together with estimate (3) gives

$$\begin{aligned}
\left|\sum_{n\le x}\mu(n)F\left(\frac{x}{n}\right)\right| &< \left|\sum_{n\le x/t}\mu(n)F\left(\frac{x}{n}\right)\right| + \left|\sum_{x/t<n\le x}\mu(n)F\left(\frac{x}{n}\right)\right| \\
&< c\left(\frac{x}{\sqrt{t}} + 2\varepsilon t^{3/2}x\right) < c\left(\sqrt{2}\varepsilon^{1/6}x + \varepsilon^{1/2}x\right) \\
&< 3c\varepsilon^{1/6}x,
\end{aligned}$$

where we used the fact that $\sqrt{2} < 2$ and $(1/2)\varepsilon^{-1/3} < t \le \varepsilon^{-1/3}$ for $\varepsilon < 1/8$. Since ε was arbitrary, we conclude that

$$\left|\sum_{n\le x}\mu(n)F\left(\frac{x}{n}\right)\right| = o(x),$$

and so $\psi(x) \sim x$. $\square$

3 Survey of the Proof of the Prime Number Theorem

Maybe it is worth spending a few minutes surveying the major players in the proof of the Prime Number Theorem. We used the Chebyshev estimates Theorem 1 of Lecture 1 to show that the Prime Number Theorem is equivalent to $\psi(x) \sim x$ (Problem 1 in Lecture 2). We also proved Abel's summation formula Proposition 1 of Lecture 2, which is a tool very important in itself. Then we introduced the Riemann Zeta function as in Definition 1 in Lecture 3 for $\sigma > 1$. An application of Abel's summation formula allowed us to write down a different formula for $\zeta(s)$ (Theorem 1 in Lecture 3) which makes sense for all $\sigma > 0$ except when $s = 1$. This function is holomorphic in the region $\sigma > 1$ with $s \neq 1$, and has no zero in this region, facts which are not hard to prove but we took them as a black box. Thus, $1/\zeta(s)$ is defined everywhere for $\sigma > 0$ (even at $s = 1$ where its value is zero) and is holomorphic in this region. The Euler product representation formula (3) of Lecture 3 allowed us to link $1/\zeta(s)$ with the Möbius function when $\sigma > 1$ and a technical result from complex analysis of Newman allowed us to conclude that this representation is true on the line $\sigma = 1$ also. This lead us to the conclusion that the series of general term $\mu(n)/n$ is convergent and its sum is zero, and now Abel's summation formula (or Problem 2 from the Homework to Lecture 2) allowed us to conclude that the summatory function of $\mu(n)$

up to x is $o(x)$ (Theorem 4 of Lecture 4). Finally, this result in conjunction with a result on the summatory function of the number of divisors function (Theorem 1 of today's lecture) were put together to finally achieve the proof of the Prime Number Theorem. Do you think that was hard? Well, if you answered yes, then you are right. And this is the easiest proof known to man of the Prime Number Theorem. There exists an *elementary* proof of the Prime Number Theorem due to Erdős and Selberg which avoids complex analysis but conceptually it is much more difficult than the chain of events described here.

4 Homework

Solve the following problems.

Problem 1. *Let $n = p_1^{\ell_1} \cdots p_k^{\ell_k}$, where $p_1, \ldots, p_k$ are distinct primes. Show that*

$$d(n) = (\ell_1 + 1) \cdots (\ell_k + 1).$$

Problem 2. *Show that*

$$\sum_{x<p\le 2x} \frac{1}{p} = O\left(\frac{1}{\log x}\right).$$

Problem 3. *Show that*

$$\sum_{x<p\le x^2} \frac{1}{p} = \log 2 + o(1)$$

as $x \to \infty$.

Problem 4. *Show that the Prime Number Theorem is equivalent to the fact that*

$$\lim_{x\to\infty} \left(\sum_{p\le x} \frac{\log p}{p} - \log x\right)$$

exists. (Hint: Use Abel's summation formula).

Problem 5. *Show that the Prime Number Theorem is equivalent to the fact that*

$$\lim_{x\to\infty} \left(\sum_{n\le x} \frac{\Lambda(n)}{n} - \log x\right)$$

exists.