Lecture 4

January 11, 2018

1 The Möbius function

One of the important functions in analytic number theory is the Möbius function, which we now introduce.

Definition 1. *The Möbius function* $\mu : \mathbb{Z}^+ \to \{-1, 0, 1\}$ *is given by* $\mu(1) = 1$, $\mu(n) = (-1)^r$ *if* n *is a product of* r *distinct primes and* $\mu(n) = 0$ *otherwise.*

In particular, $\mu(12) = 0$, $\mu(15) = 1$ and $\mu(30) = -1$. Notice that if $\sigma > 1$ then

$$\frac{1}{\zeta(s)} = \prod_{p \geq 2} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \tag{1}$$

We summarize the most important properties of the Möbius function in the following theorem. Properties (ii) and (iii) below are usually referred to as *Möbius inversion formulas.*

Theorem 1. *(i) Let* $n \in \mathbb{Z}^+$. *Then*

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

(ii) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{C}$ *and define* $F : \mathbb{R}^+ \longrightarrow \mathbb{C}$ *by*

$$F(x) = \sum_{n \leq x} f\left(\frac{x}{n}\right).$$

Then

$$f(x) = \sum_{n \leq x} \mu(n) F\left(\frac{x}{n}\right).$$

(iii) Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ and define $F : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ by

$$F(n) = \sum_{d|n} f(d).$$

Then

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Proof. (i) If $n = 1$, the result is obvious. Assume now that $n > 1$. Let $n = p_1^{\ell_1} \cdots p_r^{\ell_r}$ where $p_1, \ldots, p_r$ are distinct primes and $\ell_1, \ldots, \ell_r$ are positive integers. Let $m = p_1 \cdots p_r$. Then

$$\sum_{k|n} \mu(k) = \sum_{k|m} \mu(k).$$

But note that

$$\sum_{k|m} \mu(k) = 1 - \binom{r}{1} + \binom{r}{2} - \ldots + (-1)^r \binom{r}{r} = 0.$$

(ii) By (i),

$$\begin{aligned} f(x) &= \sum_{n \leq x} \left(\sum_{k \,|\, n} \mu(k) \right) f\left(\frac{x}{n}\right) = \sum_{k\ell \leq x} \mu(k) f\left(\frac{x}{k\ell}\right) \\ &= \sum_{k \leq x} \mu(k) \left(\sum_{\ell \leq x/k} f\left(\frac{x}{\ell k}\right) \right) = \sum_{k \leq x} \mu(k) F\left(\frac{x}{k}\right). \end{aligned}$$

(iii) Again by (i),

$$\begin{aligned} f(n) &= \sum_{c \,|\, n} \left(\sum_{d \,|\, (n/c)} \mu(d) \right) f(c) = \sum_{cd \,|\, n} \mu(d) F(c) \\ &= \sum_{d \,|\, n} \mu(d) \sum_{c \,|\, (n/d)} f(c) = \sum_{d \,|\, n} \mu(d) F\left(\frac{n}{d}\right). \end{aligned}$$

$\square$

2 A theorem of D.J. Newman

Theorem 2. *Suppose that* $a_n \in \mathbb{C}$ *with* $|a_n| \le 1$ *for* $n = 1,\ 2, \ldots$. *Form the series*

$$\sum_{n\ge 1} \frac{a_n}{n^s}. \tag{2}$$

The series converges to a function $F(s)$ *which is holomorphic in* $\mathrm{Re}(s) > 1$. *Assume that it can be extended holomorphically to* $\mathrm{Re}(s) \ge 1$. *Then series* (2) *converges to* $F(s)$ *for all complex numbers* s *with* $\mathrm{Re}(s) \ge 1$.

It is not hard to prove that $F(s)$ is holomorphic for $\mathrm{Re}(s) > 1$. In fact, not only does $F(s)$ have a derivative but in fact

$$F'(s) = -\sum_{n\ge 1} \frac{a_n \log n}{n^s} \qquad \sigma > 1.$$

We leave this as homework. It is harder to prove that the representation given by (2) converges to $F(s)$ even when $\mathrm{Re}(s) = 1$ assuming that $F(s)$ can be extended holomorphically to a domain including the line $\sigma = 1$.

3 An application of Newman's theorem

For the purposes of today's lecture, we can use Newman's theorem to prove the following.

Theorem 3.

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \tag{3}$$

Proof. For $\sigma > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

It follows that $f(s) = (s-1)\zeta(s)$ is holomorphic in the region $\sigma > 0$ and that it has no zero in the region $\sigma \ge 1$. Hence, $1/\zeta(s)$ is holomorphic in the region $\sigma \ge 1$. In fact, its formula is

$$\frac{1}{\zeta(s)} = \frac{s-1}{s\left(1-(s-1)\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx\right)} \tag{4}$$

when $\sigma \geq 1$. By Newman's Theorem 2, we have that $\sum_{n\geq 1} \frac{\mu(n)}{n}$ converges to $\frac{1}{\zeta(s)}$ for all s with $\sigma \geq 1$. In particular, it converges at $s = 1$ and now formula (4) implies the desired conclusion. $\square$

Theorem 4.

$$\sum_{n\leq x} \mu(n) = o(x). \tag{5}$$

4 Homework

Solve the following problems.

Problem 1. *Show that*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{\zeta(2)}{\zeta(4)}.$$

Can you generalize this formula?

Problem 2. *Let $(a_n)_{n\geq 1}$ be some sequence of complex numbers such that $|a_n| \leq 1$ for all $n \geq 1$. Show that the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges to a function $F(s)$ which is holomorphic for $\sigma > 1$.

Problem 3. *Let $\phi(n)$ be the Euler function which counts the number of positive integers $m \leq n$ which are coprime to n. Show that*

(i)

$$\phi(n) = \sum_{d\,|\,n} \mu(d)\frac{n}{d}.$$

(ii) Deduce from (i) that $\phi(mn) = \phi(m)\phi(n)$ whenever m and n are coprime.

(iii)

$$\sum_{d\,|\,n} \phi(d) = n.$$

Problem 4. *Let $f(X) \in \mathbb{Z}[X]$ be a non-constant polynomial with integer coefficients. Show that there exist infinitely many positive integers n such that $\mu(|f(n)|) = 0$.*

Problem 5. *A positive integer* n *is called square-full if* $p^2 \mid n$ *whenever* p *is a prime factor of* n. *Show that the formula*

$$\sum_{\substack{n \geq 1 \\ n \text{ square-full}}} \frac{1}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

is valid for all $\sigma > 1/2$. *(Hint: show that every powerful number* n *has a unique representation as* $n = u^2 v^3$, *where* u *and* v *are integers with* v *square-free).*