Lecture 3

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1 Some complex analysis

Although you might have never taken a complex analysis course, you perhaps still know what a complex number is. It is a number of the form

$$z = x + iy,$$

where x and y are reals and $i = \sqrt{-1}$. The exponential function e^z can be defined in terms of its series

$$e^z = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n \geq 0} \frac{z^n}{n!}. \tag{1}$$

More precisely, one can check, using the ratio test or the root test that the series appearing in the right hand side converges absolutely for all complex numbers z. Thus, this series defines a function which we denote by e^z. When $z = x$ is real, we recognize in the right hand side of formula (1) the Taylor expansion of the usual exponential function e^x, so it coincides with it. When $z = iy$, where y is real, we have

$$\begin{aligned} e^{iy} &= \sum_{n \geq 0} \frac{(iy)^n}{n!} = \sum_{k \geq 0} \frac{(iy)^{2k}}{(2k)!} + \sum_{k \geq 0} \frac{(iy)^{2k+1}}{(2k+1)!} \\ &= \sum_{k \geq 0} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k \geq 0} (-1)^k \frac{y^{2k+1}}{(2k+1)!} \end{aligned} \tag{2}$$

and in the right hand side of the last formula (2) we recognize the familiar Taylor expansions of $\cos y$ and $\sin y$. Thus, we get

$$e^{iy} = \cos y + i \sin y.$$

The exponential function on the real numbers has the important property that $e^{x+y} = e^x \cdot e^y$. The same is true for the complex exponential function since

$$\begin{aligned} e^{z_1+z_2} &= \sum_{n\geq 0} \frac{1}{n!}(z_1+z_2)^n = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \right) \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} = \sum_{u\geq 0} \sum_{v\geq 0} \frac{z_1^u}{u!} \frac{z_2^v}{v!} \\ &= \left(\sum_{u\geq 0} \frac{z_1^u}{u!} \right) \left(\sum_{v\geq 0} \frac{z_2^v}{v!} \right) = e^{z_1} \cdot e^{z_2}, \end{aligned}$$

where in the above calculations we used the binomial formula, the change in the order of summation $u = k,\ v = n - k$ (for all $n \geq 0$ and $0 \leq k \leq n$ whose inverse is $k = u,\ n = u + v$), as well as the fact that we can rearrange the order of the terms anyway we want since the series we are working with is absolutely convergent.

In particular,

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y).$$

If $a > 0$ is any real number, then we can use the fact that $a = e^{\log a}$ and thus define

$$a^z = e^{(\log a)z}.$$

If $z = x + iy$, then

$$a^z = e^{(\log a)(x+iy)} = e^{(\log a)x}\, e^{i(\log a)y} = a^x(\cos(a \log y) + i \sin(a \log y)).$$

In particular, $|a^z| = a^x$.

2 The Riemann Zeta Function

We are now ready to define the Riemann zeta function. We will use Riemann's notations where $s \in \mathbb{C}$ is a complex number written as $s = \sigma + it$, where σ and t are real numbers.

Definition 1. *For $s \in \mathbb{C}$ with $\sigma > 1$, we define*

$$\zeta(s) = \sum_{n\geq 0} \frac{1}{n^s}.$$

The series $\sum_{n\geq 1}\frac{1}{n^s}$ converges absolutely for $\sigma > 1$. Indeed, given $N \in \mathbb{N}$, we have

$$\left|\sum_{n=1}^{N}\frac{1}{n^s}\right| \leq \sum_{n=1}^{N}\frac{1}{|n^s|} = \sum_{n=1}^{N}\frac{1}{n^\sigma},$$

and the series

$$\zeta(\sigma) = \sum_{n\geq 1}\frac{1}{n^\sigma}$$

is convergent for all $\sigma > 1$. Furthermore, we have the Euler product representation

$$\prod_{p\geq 2}\left(1-\frac{1}{p^s}\right)^{-1} = \sum_{n\geq 1}^{s}\frac{1}{n^s}. \tag{3}$$

To see the above formula, note that

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n$$

is valid for all complex numbers $|z| < 1$ (to prove it, note that if we stop the sum at N we get $\frac{1-z^{N+1}}{1-z}$ which tends to $\frac{1}{1-z}$ when N tends to infinity because $|z| < 1$). Thus,

$$\prod_{p\geq 2}\left(1-\frac{1}{p^s}\right)^{-1} = \prod_{p\geq 2}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right),$$

and if we expand the above product, then a typical term of it is

$$\frac{1}{p_1^{\alpha_1 s}\cdots p_k^{\alpha_k s}} = \frac{1}{(p_1^{\alpha_1}\cdots p_k^{\alpha_k})^s},$$

and now the conclusion that the formula (3) holds follows from the Fundamental Theorem of Arithmetic.

Note that formula (3) allows us to give another proof that there are infinitely many primes. Indeed, assume that there are only finitely many. Then $\prod_{p\geq 2}\left(1-\frac{1}{p^s}\right)^{-1}$ is bounded as we let s tend to 1 from above on the real line. However, $\sum_{n\geq 1}\frac{1}{n}$ is divergent, which is a contradiction. This argument goes back to Euler.

3 The Riemann Zeta Function when $\sigma > 0$

In the preceding section, we defined $\zeta(s)$ for all $\sigma > 1$. In this section, we will extend this definition in a *nice* way (here, by nice we mean continuous, differentiable, analytic, etc.) to a function defined for all $s \in \mathbb{C}$ with $\sigma > 0$ except for $s = 1$.

Theorem 1. *The following formula*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{(x - \lfloor x \rfloor)}{x^{s+1}}dx \tag{4}$$

is valid for the Riemann Zeta function whenever s is a complex number with $\sigma > 1$.

Proof. Let x be any positive real number. Use Abel's summation formula with $a_n = 1$ and $f(t) = \frac{1}{t^s}$. In this case, $A(x) = \sum_{n \le x} a_n = \lfloor x \rfloor$ and we get that

$$\sum_{n \le x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s\int_1^x \frac{\lfloor u \rfloor}{u^{s+1}}du.$$

Letting $x \to \infty$, we get

$$\begin{aligned}
\zeta(s) &= 0 + s\int_1^\infty \frac{\lfloor u \rfloor}{u^{s+1}} = s\int_1^\infty \frac{(u - (u - \lfloor u \rfloor)}{u^{s+1}}du \\
&= s\int_1^\infty \frac{du}{u^s} - s\int_1^\infty \frac{(u - \lfloor u \rfloor}{u^{s+1}}du \\
&= s\left(\frac{u^{1-s}}{s-1}\Big|_{u=1}^{u=\infty}\right) - s\int_1^\infty \frac{(u - \lfloor u \rfloor)}{u^{s+1}}du \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}dx,
\end{aligned}$$

which is what we wanted to prove. □

Note that the improper integral

$$\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}dx$$

converges absolutely for all $\sigma > 0$. Thus, we may just adopt formula (4) as the definition of $\zeta(s)$, and then we see that the Riemann Zeta function can be defined for all $s \in \mathbb{C}$, $s \neq 1$ with $\sigma > 0$. In this domain, the function $\zeta(s)$ is very nice. It is continuous and in fact even differentiable everywhere in the domain $s \in \mathbb{C}$ with $\sigma > 0$. There are ways to extend it to all the complex numbers $s \neq 1$ in such a way that it remains continuous and differentiable, but we shall not need this.

Being also a function of a complex variable which is differentiable everywhere for all $s \in \mathbb{C}$ with $\sigma > 0$ with $s \neq 1$, it is in fact analytic in this domain. If you do have never seen this word, we will explain it in one of the future lectures.

You might have heard of the Riemann Hypothesis. Here it is:

Conjecture 1. *If $\zeta(s) = 0$ for some $s \in \mathbb{C}$ with $\sigma > 0$ and $s \neq 1$, then $\sigma = 1/2$.*

It is easy to see that $\overline{\zeta(s)} = \zeta(\overline{s})$. In particular, if s is real so is $\zeta(s)$. Thus, the complex zeros of $\zeta(s)$ with $\sigma > 0$ come in pairs consisting of a zero and its conjugate. Thus, it suffices to look at those ones lying in the part of the complex plane for which $t \geq 0$. The Riemann Hypothesis Conjecture 1 says that all zeros of the Riemann Zeta function with $\sigma > 0$ have $\sigma = 1/2$. It has been checked to be true for the first (i.e., those with smallest t) $1,500,000,000$ zeros. It doesn't look like it will be hard to prove does it? Well, many tried and failed. If you prove it, not only do you become famous, but you also cash in the prize of $1,000,000 offered by the Clay Mathematical Institute for a proof this conjecture (check out the web site the Clay Mathematical Institute). You get no money if you find a counterexample.

For us, the two most important properties are the following. The first one concerns the zeros of $\zeta(s)$.

Theorem 2. *The function $\zeta(s)$ has no zeros with $\sigma \geq 1$.*

The second one concerns the derivative of $\zeta(s)$, as a function of a complex variable.

Definition 2. *Let f be a function defined everywhere in some open disk $D(z;\delta) = \{w \ : \ |w - z| < \delta\}$ centered at z of radius δ. Then f has a derivative in z if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. In this case, it is denoted $f'(z)$. If f has a derivative everywhere in $D(z, \delta)$, then f is called holomorphic in $D(z;\delta)$.

Example 1. *We show here that the derivative of* e^z *exists and is itself. For this it suffices to show that for all fixed* z*, we have*

$$\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = e^z.$$

Note that

$$\frac{e^{z+h} - e^z}{h} - e^z = e^z \left(\frac{e^h - 1}{h} - 1 \right).$$

Since z *is fixed, it suffices to prove that*

$$\lim_{h\to 0} \frac{e^h - 1}{h} - 1 = 0. \tag{5}$$

Assume $|h| < 1$*. Then,*

$$\frac{e^h - 1}{h} - 1 = \frac{h}{2!} + \frac{h^2}{3!} + \cdots + \frac{h^{n-1}}{n!} + \cdots,$$

so

$$\left| \frac{e^h - 1}{h} - 1 \right| \le |h| \left(\sum_{n\ge 2} \frac{|h|^{n-2}}{n!} \right) < |h| \left(\sum_{n\ge 0} \frac{1}{n!} \right) = e|h|, \qquad |h| < 1, \tag{6}$$

which clearly implies limit (5)*.*

All rules that you know about derivatives apply here: product rule, sum rule, chain rule, etc. We shall not mention them.

We now prove that $\zeta(s)$ has a derivative for all complex numbers $s = \sigma + it \in \mathbb{C}$, $s \ne 1$ with $\sigma > 0$.

Proposition 1. *The function* $\zeta(s)$ *has a derivative for all complex numbers* $s \in \mathbb{C}$*,* $s \ne 1$ *with* $\sigma > 0$*.*

We shall not prove this.