Lecture 2

January 9, 2018

1 The Abel summation formula

The following result is very important and useful. It is called Abel's summation formula.

Proposition 1. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of complex numbers and let* $f : \mathbb{R}_{\geq 1} \longrightarrow \mathbb{C}$. *For* $x \geq 1$ *and real put*

$$A(x) = \sum_{n \leq x} a_n.$$

Assume that $f(x)$ *is continuous for* $x \geq 1$. *Then,*

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt. \tag{1}$$

Proof. Let $N = \lfloor x \rfloor$. Then,

$$\begin{aligned}\sum_{n \leq N} a_n f(n) &= A(1)f(1) + (A(2) - A(1))f(2) + \cdots + (A(N) - A(N-1))f(N) \\ &= A(1)(f(1) - f(2)) + \cdots + A(N-1)(f(N-1) - f(N)) + A(N)f(N).\end{aligned}$$

Note now that $f(i+1) - f(i) = \int_i^{i+1} f'(t)dt$ for all $i = 1, \ldots, N-1$. Furthermore, note that $A(t)$ is constant on the interval $[i, i+1)$. Thus,

$$\begin{aligned}\sum_{n \leq N} a_n f(n) &= A(N)f(N) - \sum_{i=1}^{N-1} A(i) \int_i^{i+1} f'(t)dt \\ &= A(N)f(N) - \sum_{i=1}^{N-1} \int_i^{i+1} A(t)f'(t)dt \\ &= A(N)f(N) - \int_1^N A(t)f'(t)dt.\end{aligned}$$

This proves formula (1) when $x = N$ is an integer. For the remaining values of x, note that the value of the left hand side of formula (1) does not change if we replace N by x. The value of the right hand side of it becomes, since $A(t)$ is constant on $[N, x)$,

$$\begin{aligned} A(x)f(x) &- \int_1^x A(t)f'(t)dt = A(x)f(x) - \int_N^x A(t)f'(t)dt - \int_1^N A(t)f'(t)dt \\ &= A(x)f(x) - A(N)\int_N^x f'(t)dt - \int_1^N A(t)f'(t)dt \\ &= A(x)f(x) - A(N)(f(x) - f(N)) - \int_1^N A(t)f'(t)dt \\ &= A(N)f(N) - \int_1^N A(t)f'(t)dt, \end{aligned}$$

which now completes the proof of the proposition. □

2 Useful estimates

In the remaining of the lecture, we look at some important estimates which are derived using Abel's summation formula. We start with some notation.

Here, we introduce some notations. Let $f,\ g$ be functions defined on $\mathbb{Z}_{\geq 1}$ (or on $\mathbb{R}_{>0}$) whose ranges are in the real numbers. Assume that the image of g is in $\mathbb{R}_{>0}$.

Definition 1. *(i) We say that $f(x) = O(g(x))$ if there exist constants x_0 and $c > 0$ such that $|f(x)| < cg(x)$ holds for all $x > x_0$.*

(ii) We say that $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

(iii) We say that $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

(iv) We say that $f(x) \ll g(x)$ if $f(x) = O(g(x))$.

(v) We say that $f(x) \gg g(x)$ if $g(x) \ll f(x)$.

(vi) We say that $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

Note that $f(x) \asymp g(x)$ if and only if

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

Definition 2. *Let*

$$\gamma = 1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^2}\, dt.$$

The number γ is called Euler's constant. Its numerical value is approximately $0.57721\ldots$.

It is suspected that γ is a transcendental number but it is not even known that it is irrational.

Theorem 1. *The estimate*

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \tag{2}$$

holds as $x \to \infty$.

Proof. We use Abel's summation formula Proposition 1 with the choices $a_n = 1$ and $f(t) = 1/t$. Note that

$$A(x) = \sum_{n \le x} 1 = \lfloor x \rfloor,$$

and that $f'(t) = -1/t^2$. Formula (1) now tells us that

$$\begin{aligned}
\sum_{n \le x} \frac{1}{n} &= \frac{\lfloor x \rfloor}{x} - \int_1^x \frac{\lfloor t \rfloor}{t^2}\, dt \\
&= \frac{x - (x - \lfloor x \rfloor)}{x} + \int_1^x \frac{t - (t - \lfloor t \rfloor)}{t^2}\, dt \\
&= 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{dt}{t} - \int_1^x \frac{t - \lfloor t \rfloor}{t^2}\, dt \\
&= 1 + O\left(\frac{1}{x}\right) + \left(\log t \Big|_{t=1}^{t=x}\right) - \left(\int_1^\infty \frac{t - \lfloor t \rfloor}{t^2}\, dt - \int_x^\infty \frac{t - \lfloor t \rfloor}{t^2}\, dt\right) \\
&= \log x + \gamma + O\left(\frac{1}{x}\right) + \int_x^\infty \frac{t - \lfloor t \rfloor}{t^2}\, dt \\
&= \log x + \gamma + O\left(\frac{1}{x}\right) + O\left(\int_x^\infty \frac{dt}{t^2}\right) \\
&= \log x + \gamma + O\left(\frac{1}{x}\right),
\end{aligned}$$

as $x \to \infty$, which completes the proof of the theorem. $\square$

The following function is important in analytic number theory. It is called the *von Mangold* function.

Definition 3. *For any positive integer n define $\Lambda(n)$ by the rule*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore define, for $x > 0$, the functions

$$\theta(x) = \sum_{p \leq x} \log p = \log \left(\prod_{p \leq x} p \right)$$

and

$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n).$$

Notice that

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p.$$

Also observe that since $p^2 \leq x$ is equivalent to $p \leq x^{1/2}$, $p^3 \leq x$ is equivalent to $p \leq x^{1/3}$, we get that

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots . \tag{3}$$

Note that $x^{1/m}$ appears in the above sum (3) only if $x^{1/m} \geq 2$, or, equivalently, $m \leq \log x / \log 2$. So, the above sum (3) contains $\lfloor \log x / \log 2 \rfloor$ terms. Upon noting that

$$\theta(x) \leq (\log x)\pi(x) = O(x)$$

(by Chebyshev's theorem), we get that

$$\theta(x^{1/2}) + \theta(x^{1/3}) + \ldots \leq \theta(x^{1/2})(\log x / \log 2) = O(x^{1/2} \log x),$$

so equality (3) gives

$$\psi(x) = \theta(x) + O(x^{1/2} \log x). \tag{4}$$

We now show that $\psi(x) \sim \theta(x)$. Indeed, note that if $p \geq x^{1/2}$, then $\log p \geq \log(x^{1/2}) = (\log x)/2 \gg \log x$, so

$$\begin{aligned} \theta(x) &\geq \sum_{x^{1/2} \leq p \leq x} \log p & (5) \\ &\gg (\pi(x) - \pi(\sqrt{x})) \log x & (6) \\ &= \pi(x) \log x - \pi(x^{1/2}) \log x & (7) \\ &= \pi(x) \log x + O(x^{1/2} \log x). & (8) \end{aligned}$$

Since $\pi(x) \gg x/\log x$, we get that $\theta(x) \gg x$. Comparing this with estimate (4), we get immediately that the difference $O(x^{1/2}\log x)$ between $\psi(x)$ and $\theta(x)$ is $o(\theta(x))$ as $x \to \infty$, and so

$$\psi(x) \sim \theta(x).$$

We also incidentally note that we have shown that $\pi(x) \asymp \theta(x)/\log x$.

Problem 1. *Show that $\pi(x) \asymp \psi(x)/\log x$. Deduce that in order to prove the Prime Number Theorem it suffices to show that $\psi(x) \asymp x$. The same holds if we replace $\psi(x)$ by $\theta(x)$.*

Theorem 2. *The estimate*

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1) \tag{9}$$

holds as $x \to \infty$.

Proof. We apply Abel's summation formula with $a_n = 1$ and $f(t) = \log t$. Then

$$\begin{aligned}
\sum_{n \le x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor t \rfloor}{t}\, dt \\
&= (x - (x - \lfloor x \rfloor)) \log x - \int_1^x \frac{t - (t - \lfloor t \rfloor)}{t}\, dt \\
&= x \log x + O(\log x) - \int_1^x dt + \int_1^x \frac{t - \lfloor t \rfloor}{t} dt \\
&= x \log x + O(\log x) - (x - 1) + O\left(\int_1^x \frac{dt}{t}\right) \\
&= x \log x - x + O(\log x). \qquad (10)
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\sum_{n \le x} \log n &= \log\left(\lfloor x \rfloor !\right) = \sum_{p \le x} \left(\sum_{k=1}^{\infty} \left\lfloor \frac{x}{p^k} \right\rfloor \right) \log p \\
&= \sum_{p^m \le x} \left\lfloor \frac{x}{p^m} \right\rfloor \log p = \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor \Lambda(n) \\
&= \sum_{n \le x} \frac{x}{n} \Lambda(n) - \sum_{n \le x} \left(\frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor \right) \Lambda(n) \\
&= x \sum_{n \le x} \frac{\Lambda(n)}{n} + O\left(\sum_{n \le x} \Lambda(n) \right).
\end{aligned}$$

Since
$$\sum_{n\le x}\Lambda(n)=\psi(x)=O(x),$$
we get that
$$\sum_{n\le x}\log n = x\sum_{n\le x}\frac{\Lambda(n)}{n}+O(x). \tag{11}$$
Comparing estimate (11) with estimate (10), we get
$$x\log x - x + O(\log x) = x\sum_{n\le x}\frac{\Lambda(n)}{n}+O(x),$$
and by dividing both sides by x, we get the desired estimate. $\square$

Theorem 3. *The estimate*
$$\sum_{p\le x}\frac{\log p}{p}=\log x + O(1) \tag{12}$$
holds as $x\to\infty$.

Proof. Clearly,
$$\begin{aligned}\sum_{p\le x}\frac{\log p}{p} &= \sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m}\\ &= \log x + O(1) - \sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m},\end{aligned}$$
where we used estimate (9). However, since
$$\sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m}\le \sum_p\left(\frac{1}{p^2}+\frac{1}{p^3}+\dots\right)\log p=\sum_p\frac{\log p}{p(p-1)}=O(1),$$
the desired estimate follows. $\square$

The next result is called Mertens's estimate.

Theorem 4. *There exists a constant β such that the estimate*
$$\sum_{p\le x}\frac{1}{p}=\log\log x+\beta+O\left(\frac{1}{\log x}\right) \tag{13}$$
holds as $x\to\infty$.

Proof. Define

$$a_n = \begin{cases} \frac{\log p}{p} & \text{if } n = p, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f(t) = 1/\log t$. Then $f'(t) = -1/t(\log t)^2$. Furthermore,

$$A(x) = \sum_{n \le x} a_n = \sum_{p \le x} \frac{\log p}{p} = \log x + \tau(x),$$

where $\tau(x) = O(1)$. Abel's summation formula now gives

$$\begin{aligned}
\sum_{p \le x} \frac{1}{p} &= \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t(\log t)^2}\, dt \\
&= 1 + \frac{\tau(x)}{\log x} + \int_2^x \frac{\log t + \tau(t)}{t(\log t)^2}\, dt \\
&= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{\tau(t)}{t(\log t)^2}\, dt \\
&= 1 + O\left(\frac{1}{\log x}\right) + (\log \log t)\Big|_{t=2}^{x} + \int_2^\infty \frac{\tau(t)}{t(\log t)^2}\, dt - \int_x^\infty \frac{\tau(t)}{t(\log t)^2}\, dt \\
&= \log \log x + 1 - \log \log 2 + \int_2^\infty \frac{\tau(t)}{t(\log t)^2}\, dt + O\left(\frac{1}{\log x}\right) \\
&+ O\left(\int_x^\infty \frac{dt}{t(\log t)^2}\right) \\
&= \log \log x + \beta + O\left(\frac{1}{\log x} + \left(-\frac{1}{\log t}\right)\Big|_{t=x}^{\infty}\right) \\
&= \log \log x + \beta + O\left(\frac{1}{\log x}\right),
\end{aligned}$$

where

$$\beta = 1 - \log \log 2 + \int_2^\infty \frac{\tau(t)}{t(\log t)^2}\, dt.$$

This completes the proof of the theorem. □

It can be shown that the constant β which appears in estimate (13) is given by

$$\beta = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) - \frac{1}{p}\right).$$

Can you prove that the last series appearing on the right above is convergent?

3 Homework

Solve 4 of the following problems.

Problem 2. *Assume that $(b_n)_{n\geq 1}$ is a sequence of real numbers is such that*

$$\sum_{n=1}^{\infty} \frac{b_n}{n}$$

is convergent. Show that $\sum_{n\leq x} b_n = o(x)$ as $x \to \infty$. (Hint: Apply Abel's summation formula with $a_n = b_n/n$ and $f(t) = t$).

Problem 3. *Show that there exists a constant δ such that*

$$\sum_{p\leq x} \frac{1}{p \log\log p} = \log\log\log x + \delta + O\left(\frac{1}{\log\log x}\right).$$

Problem 4. *A palindrome is a number n whose string of digits (in base 10) reads the same from left and right. For example, 151, 2332 and 12345678987654321 are all palindromes.*

(i) Let $A(x) = \{n \leq x \ : \ n \text{ is a palindrome}\}$. Show that $\#A(x) \asymp x^{1/2}$. (Hint: How many digits does a palindrome n in the vicinity of x have? How many of them determine n completely?).

(ii) Deduce from (i) that if $\alpha > 1/2$ is any fixed constant then

$$\sum_{n \text{ palindrome}} \frac{1}{n^{\alpha}}$$

is convergent while

$$\sum_{\substack{n\leq x \\ n \text{ palindrome}}} \frac{1}{n^{1/2}} \asymp \log x.$$

(Hint: Use the Abel summation formula with $a_n = 1$ if n is a palindrome and $a_n = 0$ otherwise).

Problem 5. *Prove that there exists a positive constant c_1 such that*

$$\prod_{p\leq x}\left(1 - \frac{1}{p}\right) = \frac{c_1}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

as $x \to \infty$. Do you know what c_1 is in terms of γ?

Problem 6. *Prove that there exists a positive constant c_2 such that*

$$\prod_{p \le x} \left(1 + \frac{1}{p}\right) = c_2 \log x \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

as $x \to \infty$.

Problem 7. *Generalize Problems 5 and 6 above by showing that for all fixed constants $\kappa > 0$, there exists a positive constant c_κ such that*

$$\prod_{p \le x} \left(1 + \frac{\kappa}{p}\right) = c_\kappa (\log x)^\kappa \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

as $x \to \infty$.

Note: *Obviously, if you do Problem 7, you also get credit for Problems 5 and 6.*

Problem 8. *Let $\mathcal{S}$ be the set of all positive integers of the form $2^a + b^2$ for some positive integers a and b. Show that*

$$\sum_{n \in \mathcal{S}} \frac{1}{n}$$

is convergent.

Problem 9. *Let $\mathcal{P}$ be the set of all prime numbers p that divide some Fermat number (that is, some integer of the form $2^{2^k} + 1$). Prove that*

$$\sum_{p \in \mathcal{P}} \frac{1}{p}$$

is convergent.