EXERCISES FOR THE COURSE ON ELLIPTIC CURVES

CIMPA-ICTP RESEARCH SCHOOL:
ARTIN L-FUNCTIONS, ARTIN'S PRIMITIVE ROOTS CONJECTURE AND APPLICATIONS
NESIN MATHEMATIC VILLAGE

MAY 29 - JUNE 5, 2017

Ex1 Let $y^2 = x^3 + Ax + B$ be an elliptic curve. Assume that $A, B \in Z$. Prove that any point $(x, y) \in E(\mathbb{Q})$ is of the form: $\left(\frac{a}{d^2}, \frac{b}{d^3}\right)$ with a, b and d relatively prime integers.

Ex2 Let $y^2 = (x - e_1)(x - e_2)(x - e_3)$, with $e_1, e_2, e_3 \in Z$. Suppose there exists $(x, y) \in E(Q)$ such that

$$x - e_1 = au^2 \quad x - e_2 = bv^2 \quad x - e_3 = cw^2$$

with $a, b, c \in Z$, and $u, v, w \in \mathbb{Q}$. Set

$$S = \{p \text{ prime } \mid p|(e_1 - e_2)(e_1 - e_3)(e_2 - e_3)\}$$

Then if $p|abc$ then p belongs to S.

Use this to complete the proof the weak Mordell-Weil by showing that the image of $\phi : E(\mathbb{Q}) \to \left(\mathbb{Q}^\times/\mathbb{Q}^{\times^2}\right)^3$, defined in class, is finite.