Introduction to L-functions: The Big Picture. . . we believe

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What the L?

M.N. Huxley (1992)

"What is an L-function? We know one when we see one!"

- (D) an ordinary Dirichlet series $\sum a(n)/n^s$.
  (E) an Euler product over primes.
  (F) a continuation and a functional equation.
- He noticed that some people split these up.
  (E) becomes (E) and
  (B) bounds for $a(p)$ (Ramanujan Hypothesis)
  (F) becomes (F) and
  (C) continuation for all s and
  (G) Gamma factors.
- Is there an (A) connecting them? Arithmetic? Algebraic? Analytic?
- Do ABCDEFG only occur for (H) Hermitian operator on a Hilbert space?

Iwaniec-Kowalski definition

$L(f,s)$ is an **L-function** if we have the following data and conditions:
(1) A Dirichlet series with Euler product of degree $d \geq 1$,

$$L(f,s) = \sum_{n\geq 1} \lambda_f(n) n^{-s} = \prod_p \left(1-\alpha_1(p)p^{-s}\right)^{-1} \cdots \left(1-\alpha_d(p)p^{-s}\right)^{-1}$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$, with $\lambda_f(1) = 1$, $\lambda_f(n) \in \mathbb{C}$ and $\alpha_i(p) \in \mathbb{C}$ with $|\alpha_i(p)| < p$ for all p.
(2) A gamma factor

$$\gamma(f,s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right),$$

where κ_j are either real or conjugate pairs and $\mathrm{Re}\left(\kappa_j\right) > -1$.
(3) An integer $q(f) \geq 1$, called the conductor of $L(f,s)$ such that $\alpha_i(p) \neq 0$ for $p \nmid q(f)$ and $1 \leq i \leq d$.
With all this, we have the complete L-function

$$\Lambda(f,s) = q(f)^{s/2}\gamma(f,s)L(f,s) \quad \text{with} \quad \Lambda(f,s) = \epsilon(f)\Lambda\left(\overline{f}, 1-s\right).$$

Selberg class: I

The **Selberg class** S is the set of all Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

absolutely convergent for $\mathrm{Re}(s) > 1$ satisfying the following properties.
(1) analyticity.
(2) Ramanujan conjecture.
(3) Functional equation.
(4) Euler product.

Conjecture

The Selberg class consists only of automorphic L-functions.

Selberg class: II

(1) analyticity: $(s-1)^m F(s)$ is an entire function of finite order for some non-negative integer m.
(2) Ramanujan conjecture: $a(1) = 1$ and $a(n) \ll_\epsilon n^\epsilon$ for any $\epsilon > 0$.
(3) Functional equation: put

$$\Phi(s) = Q^s \prod_{i=1}^{k} \Gamma\left(\lambda_i s + \mu_i\right) F(s) = \gamma(s) F(s)$$

where Q and the λ_i's are real and positive, the μ_i's are complex with non-negative real part, and $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then

$$\Phi(s) = \omega \overline{\Phi\left(1 - \overline{s}\right)}.$$

Selberg class: III

(4) Euler product: For $\mathrm{Re}(s) > 1$,

$$F(s) = \prod_{p, prime} F_p(s)$$

with

$$F_p(s) = \exp\left(\sum_{n=1}^{\infty} \frac{b\left(p^n\right)}{p^{ns}}\right)$$

and, for some $\theta < 1/2$,

$$b\left(p^n\right) = O\left(p^{n\theta}\right).$$

Selberg class: Examples

- Riemann zeta-function.
- Shifts $L(s + i\theta, \chi)$ of Dirichlet L-functions for primitive characters χ with $\theta \in \mathbb{R}$.
- Dedekind zeta-functions to number fields, K, with $[K : \mathbb{Q}] = n$.
- $L(s, \chi)$ with a non-primitive character χ mod q, $q \neq 1$, is **not** in $\mathcal{S}$.

Euler Products

- **(4)** Euler product: For $\mathrm{Re}(s) > 1$,

$$F(s) = \prod_{p,prime} F_p(s) \quad \text{with} \quad F_p(s) = \exp\left(\sum_{n=1}^{\infty} \frac{b\left(p^n\right)}{p^{ns}}\right)$$

and, for some $\theta < 1/2$,

$$b\left(p^n\right) = O\left(p^{n\theta}\right).$$

- The local factors, F_p, determine F.
Is there a weaker condition on the Euler Product that determines F?
- Let $F, G \in \mathcal{S}$. If, for all but finitely many primes, p, $b_F\left(p^m\right) = b_G\left(p^m\right)$ for $m = 1$ and $m = 2$, then $F = G$.

Conjecture (Strong multiplicity one)

Let $F, G \in \mathcal{S}$. If, for all but finitely many primes, p, $b_F\left(p\right) = b_G\left(p\right)$, then $F = G$.

Selberg class structure: degree (I)

Definition

The **degree** of $F \in \mathcal{S}$ is defined by

$$d_F = 2\sum_{j=1}^{f} \lambda_j.$$

Conjecture (Degree Conjecture)

For all $F \in \mathcal{S}$, $d_F \in \mathbb{Z}$.

Kaczorowski, Perelli (2011): true for $0 \leq d_F \leq 2$.
Somewhat stronger:

Conjecture (Strong λ Conjecture)

Let $F \in \mathcal{S}$. All λ_j appearing in the gamma-factors of the functional equation can be chosen to be equal to $1/2$.

Selberg class structure: degree (II)

- Examples:

  degree 0: constant function, $F(s) = 1$.

  degree 1: the Riemann zeta-function
  shifts $L(s + i\theta, \chi)$ of Dirichlet L-functions for primitive characters ξ with $\theta \in \mathbb{R}$.

  degree n: Dedekind zeta-functions to number fields, K, with $[K : \mathbb{Q}] = n$.

Theorem (Riemann-von Mangoldt Formula)

If $N_F(T)$ count (with multiplicities) the number of zeros of $F \in \mathcal{S}$ in the rectangle $0 \leq \mathrm{Re}(s) \leq 1$ with $|\mathrm{Im}(s)| \leq T$. Then

$$N_F(t) = \frac{d_F}{\pi} T \log(T) + O(T).$$

Selberg class structure: conductor

Definition

The **conductor** of $F \in \mathcal{S}$ is defined by

$$q_F = (2\pi)^{d_F} Q^2 \prod_j \lambda_j^{2\lambda_j}.$$

- It provides some finer structure to functions of the same degree.

Conjecture

For all $F \in \mathcal{S}$, $q_F \in \mathbb{Z}$.

- Examples:

  degree 1: $q_{\zeta(s)} = 1$ and $q_{L(s,\chi)}$ is the modulus of χ, if χ is primitive.

  degree n: $q_{\zeta_K(s)} = |d_K|$.

Selberg class structure: primitive elements

- $\mathcal{S}$ is multiplicatively closed.
- A function $F \in \mathcal{S}$ is called **primitive** if $F = F_1 F_2$ with $F_1, F_2 \in \mathcal{S}$ implies that $F_1 \equiv 1$ or $F_2 \equiv 1$.
- Every $F \in \mathcal{S}$ can be factored as a product of primitive elements.

Conjecture

Factorisation into primitives is unique in $\mathcal{S}$.

- Examples:
  any element of the Selberg class of degree one is primitive.
  Dedekind zeta-functions for cyclotomic fields ($\neq \mathbb{Q}$) are not primitive.

Selberg class structure: orthogonality

- Recall
$$\sum_{p\leq x} 1/p = \log\log(x) + O(1).$$

Conjecture (SOC)

For any primitive functions F_1 and F_2,
$$\sum_{p\leq x} \frac{a_{F_1}(p)\overline{a_{F_2}(p)}}{p} = \begin{cases} \log\log(x) + O(1) & \text{if } F_1 = F_2, \\ O(1) & \text{otherwise.} \end{cases}$$

SOC implies the following:

- ζ is the only primitive function in $\mathcal{S}$ with a pole at $s = 1$
- strong multiplicity one conjecture
- unique factorisation
- $F(s) \neq 0$ for $\mathrm{Re}(s) \geq 1$
- Artin's conjecture

G(rand) RH

Conjecture (Grand Riemann hypothesis (GRH))

The nontrivial zeros of members of the Selberg class lie on the critical line $1/2 + it$ with t a real number.

- The restriction to nontrivial zeros is important, because with $r_1 + r_2 - 1 > 0$, we saw that for $\zeta_K(0) = 0$.
- Functional equation, but no Euler product:
$$L(s) = \frac{1 - i\alpha}{2} L(s, \chi) + \frac{1 + i\alpha}{2} L(s, \overline{\chi})$$
where $\alpha = \left(\sqrt{10 - 2\sqrt{5}} - 2\right) / \left(\sqrt{5} - 1\right)$.
- Euler product, but with $\theta = 1/2$ allowed:
$$\left(1 - 2^{1-s}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$
- No Ramanujan hypothesis. Let χ be an odd primitive character. $G(s) = L(2s - 1/2, \chi)$ and $F(s) = G(s - \delta)G(s + \delta)$ for $\delta \in (0, 1/4)$.

Zeta functions and modular forms (I)

- Michel mentioned connection between Riemann zeta function and $\theta_0(iz)$ for the Jacobi θ function

$$\theta_0(z) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2 z\right).$$

- If $f(z) = \sum_{n=0}^{\infty} a(n)\exp(2\pi i n z/\lambda)$ is a modular form, its L-function is

$$L(f,s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

- For $\theta_0(z)$, $\lambda = 2$, $a_n = 1$ if n square and $a_n = 0$ otherwise. Then

$$L\left(\theta_0, s\right) = \sum_{n=1}^{\infty} a_n n^{-s} = \zeta(2s).$$

Key Point (Hecke's Converse Theorem (1936))

We have a 1-to-1 correspondence between modular forms from the "full modular group" with a growth condition and Dirichlet series.

Zeta functions and modular forms (II)

- But there are other groups too. The congruence subgroups.
- Weil (1967) found a converse theorem for these.

Theorem (Modularity)

For any elliptic curve E over $\mathbb{Q}$, there exists a newform f of weight 2 for some congruence subgroup $\Gamma_0(N)$ such that $L_E(s) = L(f,s)$.

- E.g., $E : Y^2 = X^3 - X$. $a_p = p + 1 - |E(\mathbb{Z}/p\mathbb{Z})|$.
  $a_5 = -2$, $a_9 = -3$, $a_{13} = 6, \ldots$, so

$$f(z) = q - 2q^5 - 3q^9 + 6q^{13} + \cdots, \quad q = \exp(2\pi iz).$$

  This is a newform of weight 2 for the congruence subgroup $\Gamma_0(32)$.
  $L_E(s) = L(f,s)$.

Zeta functions and modular forms (III)

So, we have two things . . .

- Arithmetic L-functions
  –described by Euler products,
  –arithmetic meaning is clear,
  –analytic properties are conjectural.
- Automorphic L-functions
  –described by Dirichlet series,
  –analytic properties are clear,
  –Euler product and arithmetic meaning are more mysterious.

Langlands Program

No!
There is just one thing. They are both the same.