Introduction to L-functions: Dedekind zeta functions

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Dedekind zeta function

Definition

Let K be a number field. We define for $\mathrm{Re}(s) > 1$ the **Dedekind zeta function** $\zeta_K(s)$ of K by the formula

$$\zeta_K(s) = \sum_{\mathfrak{a}} \left(\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})\right)^{-s},$$

where the sum is over all non-zero integral ideals, $\mathfrak{a}$, of $\mathcal{O}_K$.

- Euler product exists:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \left(\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})\right)^{-s}\right)^{-1},$$

where the product extends over all prime ideals, $\mathfrak{p}$, of $\mathcal{O}_K$.

$\mathrm{Re}(s) > 1$

Proposition

For any $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$, $\zeta_K(s)$ converges absolutely.

Proof:

$$|\zeta_K(s)| = \left| \prod_{\mathfrak{p}} \left(1 - \left(\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})\right)^{-s}\right)^{-1} \right| \leq \prod_{p} \left(1 - \frac{1}{p^{\sigma}}\right)^{-n} = \zeta(\sigma)^n,$$

since there are at most $n = [K : \mathbb{Q}]$ many primes $\mathfrak{p}$ lying above each rational prime p and $\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p}) \geq p$.

A reminder of some algebraic number theory

- If $[K : \mathbb{Q}] = n$, we have n embeddings of K into $\mathbb{C}$. r_1 embeddings into $\mathbb{R}$ and $2r_2$ embeddings into $\mathbb{C}$, where $n = r_1 + 2r_2$. We will label these $\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}$.
- If $\alpha_1, \ldots, \alpha_n$ is a basis of $\mathcal{O}_K$, then

$$d_K = \left(\det\left(\sigma_i\left(\alpha_j\right)\right)\right)^2.$$

- Units in $\mathcal{O}_K$ form a finitely-generated group of rank $r = r_1 + r_2 - 1$. Let $u_1, \ldots, u_r$ be a set of generators. For any embedding σ_i, set $N_i = 1$ if it is real, and $N_i = 2$ if it is complex. Then

$$R_K = \det\left(N_i \log\left|\sigma_i\left(u_j\right)\right|\right)_{1 \le i,j \le r}.$$

  w_K is the number of roots of unity contained in K.
- The ideal class group, I_K, is the quotient group J_K/P_K, with J_K the group of fractional ideals of $\mathcal{O}_K$, P_K its subgroup of principal ideals. Class number, h_K, is the size of the ideal class group.

Functional equation

- Riemann: $\xi(s) = \pi^{-s/2} s(s-1)\Gamma(s/2)\zeta(s)$ is entire and
$$\xi(s) = \xi(1-s).$$
- The complete zeta function
$$\Lambda_K(s) = \underbrace{\left(\frac{|d_K|}{4^{r_2}\pi^n}\right)^{s/2} \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)}_{\text{local factor at infinity}} \prod_{\mathfrak{p}} \underbrace{\left(1 - \left(\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})\right)^{-s}\right)^{-1}}_{\text{local factor at } \mathfrak{p}}.$$
- Then
$$\Lambda_K(s) = \Lambda_K(1-s).$$
- Unlike ξ, Λ_K has two simple poles at $s = 0$ and 1.

Theorem

$\zeta_K(s)$ has an analytic continuation to $\mathbb{C}\backslash\{1\}$ with a simple pole at $s = 1$.

Special Values: $s = 1$ (what does 1 mean?)

- The residue at $s = 1$ of the Riemann zeta function is 1.
- Class number formula:

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\sqrt{|d_K|} w_K}.$$

- Proved by Dedekind.
- This should remind you of $L(s, \chi)$, when χ is a real primitive character. There's a reason why...

Class Number Formula I

- Class number formula:

$$\mathrm{Res}_{s=1}\left(\zeta_K(s)\right) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\sqrt{|d_K|}\, w_K}.$$

- Residue of a function on complex plane: purely analytic
  Right-hand side: purely arithmetic

Principle

Analytic objects can encode arithmetic information.

Class Number Formula II

- Class number formula:

$$\mathrm{Res}_{s=1}\left(\prod_{\mathfrak{p}}\left(1-\left(\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})\right)^{-s}\right)^{-1}\right)=\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{\sqrt{|d_K|}w_K}.$$

- "Local" objects on left-hand side
  "Global" information on right-hand side

Local-global principle

Local objects can encode global information.

Special Values: $s = 0$

- $s = 0$: a zero of order r (the rank of the unit group of $\mathcal{O}_K$).

$$\lim_{s \to 0} s^{-r} \zeta_K(s) = -\frac{h_K R_K}{w_K}.$$

Special Values: Riemann recap

- Bernoulli numbers, B_k, defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42, \ldots$

- $$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

- $$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}.$$

Note: $\zeta(-2k) = 0$ since $B_{2k+1} = 0$ for $k \geq 1$.

Special Values: s, a negative integer

- $\zeta_K(s)$ vanishes at all negative even integers.
- If K is not totally real (i.e., $r_2 \neq 0$), then $\zeta_K(-(2k+1)) = 0$.
- If K is totally real (i.e., $r_2 = 0$), then $\zeta_K(-(2k+1)) \in \mathbb{Q}^*$.
  In fact, $\zeta_K(2k) \in \pi^{2nk}\mathbb{Q}/\sqrt{D}$.
- From functional equation:

$$\mathrm{ord}_{s=-k}\zeta_K(s) = \begin{cases} r_1 + r_2 & \text{if } k \text{ is even} \\ r_2 & \text{if } k \text{ is odd.} \end{cases}$$

- $\zeta_K(-k) =$ K-theory quantities, higher regulators,...

Prime Ideal Theorem

Theorem (Landau, 1903)

Let $\pi_K(x) = \left|\{\mathfrak{p} \subset \mathcal{O}_K \text{ prime} : N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x\}\right|$*. We have*

$$\pi_K(x) \sim \frac{x}{\log(x)}.$$

- Surprising that the coefficient is 1 (no arithmetic of K!).
- Proof: via $\psi_K(x) = \sum_{N(\mathfrak{a}) \leq x} \Lambda_K(\mathfrak{a})$, where

$$\Lambda_K(\mathfrak{a}) = \begin{cases} \log N(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^k, \\ 0 & \text{otherwise.} \end{cases}$$

- Use

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} + \text{higher terms.}$$

- $\zeta_K(s)$ has no other zeros or poles with $\mathrm{Re}(s) = 1$.

ERH

Conjecture (Extended Riemann hypothesis (ERH))

The nontrivial zeros of the Dedekind zeta function of any algebraic number field lie on the critical line: $\mathrm{Re}(s) = 1/2$.

- $[K : \mathbb{Q}] = d$, $D = |\mathrm{disc}(K)|$ and $c > 0$. Then $\zeta_K(s)$ has no zero with
$$\mathrm{Re}(s) \geq 1 - \frac{c}{d^2 \log\left(D(|t| + 3)^d\right)},$$
except possibly a simple real zero $s < 1$.
- $K = \mathbb{Q}(\zeta_p)$: at most one zero (necessarily simple and real) satisfying
$$\mathrm{Re}(s) \geq 1 - \frac{c}{\log\left(p(|t| + 3)\right)}.$$
- Stark: If K has no quadratic subfield, then $\zeta_K(s)$ has no exceptional zero.

E.g., Quadratic Number Field

D squarefree, $K = \mathbb{Q}\left(\sqrt{D}\right)$ and $\chi_d(m) = (d_K/m)$.

- p an odd prime
  inert: $(p) = \mathfrak{p}$, if $(d_K/p) = -1$,
  ramified: $(p) = \mathfrak{p}^2$, if $(d_K/p) = 0$,
  split: $(p) = \mathfrak{p}_1\mathfrak{p}_2$, if $(d_K/p) = 1$.
- $p = 2$
  inert: $(p) = \mathfrak{p}$, if $D \equiv 5 \bmod 8$
  ramified: $(p) = \mathfrak{p}^2$, if $D \equiv 2, 3, 6, 7 \bmod 8$,
  split: $(p) = \mathfrak{p}_1\mathfrak{p}_2$, if $D \equiv 1 \bmod 8$.

$$\begin{aligned}
\zeta_K(s) &= \prod_{(d/p)=1} \left(1 - p^{-s}\right)^{-2} \prod_{(d/p)=0} \left(1 - p^{-s}\right) \\
&\quad \times \prod_{(d/p)=-1} \left(1 - p^{-s}\right)^{-1} \left(1 + p^{-s}\right)^{-1} \\
&= \zeta(s) L\left(s, \chi_d\right).
\end{aligned}$$

Dirichlet Characters

- Given a finite group, X, of Dirichlet characters, we can associate a number field, K, to X.

  (1) $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_m\right)/\mathbb{Q}\right) \cong \left(\mathbb{Z}/m\mathbb{Z}\right)^{\times}$.
  So a Dirichlet character mod m acts on $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_m\right)/\mathbb{Q}\right)$. I.e.,

  $$\chi : \mathrm{Gal}\left(\mathbb{Q}\left(\zeta_m\right)/\mathbb{Q}\right) \to \mathrm{GL}_1(\mathbb{C}) \quad \text{– a 1-dimensional Galois representation.}$$

  (2) Let $n = \mathrm{lcm}_{\chi \in X} f_\chi$.
  So X is a subgroup of the characters of $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_n\right)/\mathbb{Q}\right)$.

  (3) $H = \cap_{\chi \in X} \ker \chi$. K be the fixed field of H.
  Furthermore, $X \cong \mathrm{Gal}(K/\mathbb{Q})$.

- Kronecker-Weber Theorem:
  all abelian extensions of $\mathbb{Q}$ lie inside cyclotomic fields.

- So for any abelian extension, K, of $\mathbb{Q}$, we can associate a finite group of Dirichlet characters, X_K.

Zeta Function Factorisation

Theorem

Let X be a group of Dirichlet characters, K the associated field, and $\zeta_K(s)$ the Dedekind zeta function of K. Then

$$\zeta_K(s) = \prod_{\chi \in X} L(\chi, s).$$

Corollary

If K is an abelian extension of $\mathbb{Q}$, then $\zeta_K(s)/\zeta(s)$ is an entire function.

Proof of theorem: compare the Euler factors for p on each side. Note that since K is a Galois extension

$$(p) = (\mathcal{P}_1 \cdots \mathcal{P}_g)^e$$

and each $\mathcal{P}_i$ has residue class degree f.

A taste of some less simple algebraic number theory

Let K/k be a Galois extension of number fields with Galois group G.

- For each prime ideal $\mathfrak{p}$ of k, let $\mathfrak{P}$ be a prime ideal in K over $\mathfrak{p}$. Let $D_{\mathfrak{P}} = \{\sigma \in G : \sigma(\mathfrak{P}) = \mathfrak{P}\}$ be the **decomposition group of** $\mathfrak{P}$.
- There is a surjective homomorphism $D_{\mathfrak{P}} \to \mathrm{Gal}\left((K/\mathfrak{P})/(k/\mathfrak{p})\right)$.
  Its kernel, $I_{\mathfrak{P}}$, is the **inertia group of** $\mathfrak{P}$.
  If $\mathfrak{P}$ is unramified over $\mathfrak{p}$, then $I_{\mathfrak{P}}$ is trivial.
- We have a Frobenius element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}/I_{\mathfrak{P}}$ which is the inverse image of the Frobenius element of $\mathrm{Gal}\left((K/\mathfrak{P})/(k/\mathfrak{p})\right)$.
  If $\mathfrak{P}$ is unramified over $\mathfrak{p}$, then $\sigma_{\mathfrak{P}}$ is a single element in $D_{\mathfrak{P}}$.
- Since $\mathrm{Gal}(K/k)$ is transitive on primes $\mathfrak{P}$ lying over $\mathfrak{p}$, all the Frobenius elements $\sigma_{\mathfrak{P}}$ for $\mathfrak{P}$ over $\mathfrak{p}$ are conjugate.
  If $\mathrm{Gal}(K/k)$ is abelian, this conjugacy class for $\mathfrak{p}$ contains one single element, the **Artin symbol**.

Artin L-functions

- Let K/k be a Galois extension of number fields, V a finite-dimensional $\mathbb{C}$-vector space and (ρ, V) be a representation of $\mathrm{Gal}(K/k)$.
- (unramified) If $\mathfrak{p} \subset k$ is unramified in K and $\mathfrak{p} \subset \mathfrak{P} \subset K$, put

$$L_{\mathfrak{p}}(s,\rho) = \det{}^{-1}\left(I_V - N_{k/\mathbb{Q}}(\mathfrak{p})^{-s}\rho\left(\sigma_{\mathfrak{P}}\right)\right).$$

  Depends only on conjugacy class of $\sigma_{\mathfrak{P}}$ (i.e., only on $\mathfrak{p}$), not on $\mathfrak{P}$.
- (general) If G acts on V and H subgroup of G, then

$$V^H = \{v \in V : h(v) = v, \forall h \in H\}.$$

  With $\rho|_{V^{I_{\mathfrak{P}}}} : \mathrm{Gal}(K/k) \to GL\left(V^{I_{\mathfrak{P}}}\right)$.

$$L_{\mathfrak{p}}(s,\rho) = \det{}^{-1}\left(I - N_{k/\mathbb{Q}}(\mathfrak{p})^{-s}\rho|_{V^{I_{\mathfrak{P}}}}\left(\sigma_{\mathfrak{P}}\right)\right).$$

Definition

For $\mathrm{Re}(s) > 1$, the **Artin L-function** belonging to ρ is defined by

$$L(s,\rho) = \prod_{\mathfrak{p} \subset k} L_{\mathfrak{p}}(s,\rho).$$

Artin L-functions: Properties

(1) $L(s, \rho)$ converges absolutely and uniformly for $\mathrm{Re}(s) > 1$.
(2) If (ρ, V) is the trivial representation, then

$$L(s, \rho) = \zeta_K(s).$$

(3) If $\chi_\rho : \mathrm{Gal}(K/k) \to \mathbb{C}$ is the character of (ρ, V), then

$$L(s, \rho) = L\left(s, \chi_\rho\right).$$

(4) If ρ_1 and ρ_2 are representations with characters χ_1 and χ_2,

$$L\left(s, \chi_1 + \chi_2\right) = L\left(s, \chi_1\right) L\left(s, \chi_2\right).$$

(5) If H is a subgroup of G, χ is a character of H and $\mathrm{Ind}(\chi)$ is the character of G induced from χ

$$L(s, \mathrm{Ind}(\chi)) = L\left(s, \chi\right).$$

Artin's Conjecture

Conjecture (Artin's Conjecture)

If ρ is a non-trivial irreducible representation, then $L(s, \rho)$ has an analytic continuation to the whole complex plane.

- We can prove meromorphic.
- Proof.
  (1) Use Brauer's Theorem:

$$\chi = \sum_i n_i \mathrm{Ind}\left(\chi_i\right),$$

with χ_i one-dimensional characters of subgroups and $n_i \in \mathbb{Z}$.
(2) Use Properties (4) and (5).
(3) $L\left(s, \chi_i\right)$ is meromorphic (Hecke L-function).