Introduction to L-functions: Dirichlet L-functions

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Objectives

- Theory of zeta functions and L-functions
- Their use in proof of known Artin primitive roots results
- Tying them together with other lectures in this research school.

Varieties of zeta functions

- You can associate a zeta function to almost any mathematical object. Don't believe me?

Varieties of zeta functions

- From number theory:
  Dedekind zeta function of a number field
  Epstein zeta function of a quadratic form
  Goss zeta function of a function field
  p-adic zeta function
- From algebraic geometry:
  Local zeta-function of a characteristic p variety
  Hasse-Weil L-function of a variety
  Motivic zeta function
- Automorphic L-functions of cusp forms on $GL(m)$.

Varieties of zeta functions

Still more...

- From analysis:
  Selberg zeta-function of a Riemann surface
  Witten zeta function of a Lie group
  Spectral zeta function of an operator
- Dynamical systems:
  Artin-Mazur zeta-function
  Ruelle zeta function
- Ihara zeta-function of a graph
- Airy zeta function, related to the zeros of the Airy function

Generalisations of zeta functions

- You can generalise the Riemann zeta function in many ways too.
- Hurwitz zeta function: for $0 < a \leq 1$,

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

- multiple zeta functions:

$$\zeta(s_1, s_2, \ldots, s_t) = \sum_{n_1 > n_2 > \cdots > n_t > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_t^{s_t}}.$$

  Many striking relations between them. E.g.,

$$\zeta(2,1) = \zeta(3) \quad \text{and} \quad \zeta(2,4) = (13/3)\zeta(5,1) + (7/3)\zeta(3,3).$$

- "twist" it using characters

Dirichlet Series

- Riemann:
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$
- Dirichlet series:
$$\sum_{n=1}^{\infty} a_n n^{-s},$$
where $a_n \in \mathbb{C}$.
- Examples:
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s},$$
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \phi(n) n^{-s},$$
$$\cdots$$

Why?

- Starts with Euler, arithmetic function $f : \mathbb{N} \to \mathbb{C}$:

$$E_f(z) = \sum_n f(n) z^n,$$
$$D_f(s) = \sum_n f(n) n^{-s}.$$

- $E_f(z)$: useful for additive problems.
  E.g., $p(n)$, the partition function

$$\sum_n p(n) z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-1}.$$

- $D_f(s)$: useful for multiplicative functions.

Key Point

Euler was the first to mix analysis with arithmetic.

Dirichlet Characters: Definitions

- A, a finite abelian group
  a group character χ is a homomorphism from A to $\mathbb{C}^*$.
- $A = (\mathbb{Z}/m\mathbb{Z})^*$, the invertible elements of $\mathbb{Z}/m\mathbb{Z}$.
  Dirichlet characters mod m:
  (1) $\chi(kn) = \chi(k)\chi(n)$ for all $k, n \in (\mathbb{Z}/m\mathbb{Z})^*$.
  Extend domain of χ to $\mathbb{Z}$ by
  (2) $\chi(m+n) = \chi(n)$ for all $n \in \mathbb{Z}$,
  (3) $\chi(n) = 0$ iff $\gcd(m, n) \neq 1$.
- Terminology:
  χ has period m. The **conductor**, f_χ, is the smallest period of χ.
  If $m = f_\chi$, χ is called **primitive**.
  If $f_\chi = 1$ (i.e., $\chi(n) = 1$ for all n), χ is called **principal**.
  If $\chi(n) \in \mathbb{R}$ for all n, χ is called **real**.
  If $\chi(-1) = 1$, χ is called **even**. If $\chi(-1) = -1$, χ is called **odd**.

Dirichlet Characters: Examples

- An example you know.
  Let p is an odd prime number, then the Legendre symbol,
  $\chi(n) = (n/p)$ is a primitive (and real) Dirichlet character mod p.
- Let $\chi : (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{C}^*$ be defined by $\chi(1) = \chi(5) = 1$,
  $\chi(3) = \chi(7) = -1$.
  Notice $\chi(a+4) = \chi(a)$, so χ may be defined mod 4.
  Since 4 is minimal, $f_\chi = 4$.

Dirichlet L-function

Definition

Let χ be a Dirichlet character modulo m. For $\mathrm{Re}(s) > 1$, we define the **Dirichlet L-function** associated to χ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

- Introduced by Dirichlet (1837) to prove that there are infinitely many primes in arithmetic progressions.

  Properties:
- analytic for $\mathrm{Re}(s) > 1$
- Euler product:
  $$L(s,\chi) = \prod_{p} \left(1 - \chi(p) p^{-s}\right)^{-1}.$$

Functional Equation (I)

- Riemann: $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire and

$$\xi(s) = \xi(1-s).$$

- χ a primitive character modulo m. **Complete L-function**

$$\Lambda(s,\chi) = \underbrace{\left(\frac{\pi}{m}\right)^{-s/2}\Gamma\left(\frac{s+\delta}{2}\right)}_{\text{local factor at infinity}} \prod_p \underbrace{\left(1-\chi(p)p^{-s}\right)^{-1}}_{\text{local factor at } p},$$

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

- $\Lambda(s,\chi)$ is an entire function, if $m > 1$.
  $\Gamma(s)$ has simple poles for all $s \in \mathbb{Z}$ with $s \le 0$.
  Thus $L(s,\chi)$ has simple zeroes when $(s+\delta)/2 \in \mathbb{Z}$ with $(s+\delta)/2 \le 0$.
  These are the **trivial zeroes** of $L(s,\chi)$.

Functional Equation (II)

- Put $\tau(\chi) = \sum_{n=1}^{m} \chi(n) \exp(2\pi i n/m)$, then

$$\Lambda(1-s, \overline{\chi}) = \frac{i^{\delta} m^{1/2}}{\tau(\chi)} \Lambda(s, \chi).$$

- $\tau(\chi)$ is called the **Gauss sum** associated to χ.
  We have $|\tau(\chi)| = m^{1/2}$, so **root number** of $L(s, \chi)$ satisfies

$$\left| \frac{i^{\delta} m^{1/2}}{\tau(\chi)} \right| = 1.$$

Theorem

If χ is a primitive mod m character and non-trivial, then $L(s, \chi)$ can be extended analytically to an entire function.

Special Values: Riemann recap

- Bernoulli numbers, B_k, defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42, \ldots$

- For $k \geq 1$,

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

- For $k \geq 0$,

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}.$$

Note: for $k \geq 1$, $B_{2k+1} = 0$, so $\zeta(-2k) = 0$.

Special Values: s, a negative integer

- χ a primitive character mod m. Generalised Bernoulli numbers, $B_{k,\chi}$:

$$\sum_{a=1}^{m} \frac{\chi(a) x e^{ax}}{e^{mx}-1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{x^k}{k!}.$$

  $m = 1$: $B_{1,\chi} = 1/2 = -B_1$ and $B_{n,\chi} = B_n$ otherwise.

- $$L(-k,\chi) = -\frac{1}{k+1} B_{k+1,\chi}.$$

  Note that $B_{k,\chi} = 0$, if $k \not\equiv \delta \bmod 2$ (unless $k = m = 1$).

- zeros:
  If χ is a primitive character with $\chi(-1) = 1$, then the only zeros of $L(s,\chi)$ with $\mathrm{Re}(s) < 0$ are at the negative even integers.
  If χ is a primitive character with $\chi(-1) = -1$, then the only zeros of $L(s,\chi)$ with $\mathrm{Re}(s) < 0$ are at the negative odd integers.

Special Values: $s = 1$

Let χ be a primitive character mod m. Put $\zeta_m = \exp(2\pi i/m)$.

$$L(1,\chi) = \begin{cases} \pi i \dfrac{\tau(\chi)}{m} B_{1,\overline{\chi}} = \pi i \dfrac{\tau(\chi)}{m} \dfrac{1}{m} \displaystyle\sum_{a=1}^{m} \overline{\chi}(a) a & \text{if } \chi(-1) = -1, \\ -\dfrac{\tau(\chi)}{m} \displaystyle\sum_{a=1}^{m} \overline{\chi}(a) \log|1 - \zeta_m^a| & \text{if } \chi(-1) = 1. \end{cases}$$

Let χ be a primitive real character mod m.
Put $K = \mathbb{Q}\left(\sqrt{\chi(-1)m}\right)$, h is its class number, w is the number of roots of unity in it and ϵ is its fundamental unit.

$$L(1,\chi) = \begin{cases} \dfrac{2\pi h}{w\sqrt{m}} & \text{if } \chi(-1) = -1, \\ \dfrac{2h \log|\epsilon|}{\sqrt{m}} & \text{if } \chi(-1) = 1. \end{cases}$$

GRH

Conjecture (Generalised Riemann hypothesis (GRH))

For every Dirichlet character χ *and every complex number* s *with* $L(s, \chi) = 0$, *if* $0 \leq \mathrm{Re}(s) \leq 1$, *then* $\mathrm{Re}(s) = 1/2$.

- First stated by Adolf Piltz in 1884.
- Numerically:
  Rumely (1996)
  Platt (2013): for all primitive characters of modulus $m \leq 400,000$.
  $|\mathrm{Im}(s)| \leq \max\left(10^8/m, 7.5 \cdot 10^7/m + 200\right)$ when m is even
  $|\mathrm{Im}(s)| \leq \max\left(10^8/m, 3.75 \cdot 10^7/m + 200\right)$ when m is odd.

Theorem (Helfgott, 2013)

Ternary Goldbach conjecture: every odd number greater than 5 *can be written as the sum of precisely three primes.*

GRH: Theoretical Results

- Dirichlet (1837): $L(1, \chi) \neq 0$.
  Implies infinitely many primes of form $an + b$ where $(a, b) = 1$.
- Let $\delta > 0$ and χ be a non-real character mod m. Then

$$\mathrm{Re}(s) < 1 - \frac{\delta}{\log(m(2 + |\mathrm{Im}(s)|))}.$$

- $\delta > 0$, χ primitive real mod m. If $|\mathrm{Im}(s)| \geq \delta / \log(m)$, then

$$\mathrm{Re}(s) < 1 - \frac{\delta}{5 \log(m(2 + |\mathrm{Im}(s)|))}.$$

- Siegel-Landau zeros:
  at most one zero with $\mathrm{Re}(s) > 1 - \delta / \log(m)$ and $|\mathrm{Im}(s)| < \delta / \log(m)$.
  Such a zero, s, is necessarily real and simple.For any $\epsilon > 0$,

$$s \leq 1 - \frac{c(\epsilon)}{m^{\epsilon}}$$

Prime Number Theorem for arithmetic progressions

$$\psi(x; m, a) = \sum_{n \leq x, n \equiv a \bmod m} \Lambda(n).$$

Recall

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Siegel-Walfisz)

Let N be any positive constant. Then there exists a positive number $C(N)$, such that if $m \leq (\log x)^N$, then

$$\psi(x; m, a) = \frac{x}{\phi(m)} + O\left(x \exp\left(-C(N)(\log(x))^{1/2}\right)\right),$$

uniformly in m.