Introduction to L-functions Exercises
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1 Lecture 1: Dirichlet L-functions

(1) Suppose that $0 < \alpha \leq 1$. For $\mathrm{Re}(s) > 1$, we define the *Hurwitz zeta function* by the formula

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}.$$

(a) Show that $\zeta(s, 1/2) = (2^s - 1)\,\zeta(s)$.
(b) If χ is a character mod k

$$L(s, \chi) = k^{-s} \sum_{r=1}^{k} \chi(r) \zeta(s, r/k).$$

(2) Let G be a finite Abelian group.
(a) With multiplication of characters is defined by the relation $(\chi_1 \chi_2)(g) = \chi_1(g)\chi_2(g)$, show that the set of characters, $\widehat{G}$, of G forms an abelian group of order n.
What is the identity element of $\widehat{G}$?
What is the inverse of χ in terms of χ?
(b) Prove that $G \cong \widehat{G}$.
(c) Show that

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi \text{ is trivial (i.e., } \chi(g) = 1 \text{ for all } g \in G, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Show that

$$\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |G| & \text{if } g \text{ is the identity in } G, \\ 0 & \text{otherwise.} \end{cases}$$

2 Lecture 2: Dedekind zeta functions

(1) Using the sketch from the lecture, complete the proof of the following statement:
Let X be a group of Dirichlet characters, K the associated field, and $\zeta_K(s)$ the Dedekind zeta function of K. Then

$$\zeta_K(s) = \prod_{\chi \in X} L(\chi, s).$$

If K is an abelian extension of $\mathbb{Q}$, then $\zeta_K(s)/\zeta(s)$ is an entire function.

(2) Let $\chi(n) = (5/n)$, the Kronecker symbol.
(a) Using a software package like Pari (the Ser() function will help), or packages like Sage or Maple, calculate $B_{k,\chi}$ for $k = 0, \ldots, 6$ and hence determine $L(-k, \chi)$ for $k = 0, \ldots, 5$.
(b) Using the factorisation of $\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ and part (a), compute $\zeta_{\mathbb{Q}(\sqrt{5})}(-k)$ for $k = 0, \ldots, 5$.

(3) Let $\chi(n)$ be defined modulo 4 by $\chi(1) = 1$ and $\chi(3) = -1$.
The Galois group, G, of $\mathbb{Q}(i)$ is $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to the group of characters generated by χ.
Let $V = \mathbb{C}$ and consider (χ, V) as a one-dimensional representation of G.
Using the definition of the local factors of the Artin L-function, show that for $p = 5$, the Artin local factor, $L_5(s, \chi)$ is equal to the local factor for $p = 5$ that arises in the Euler product of the Dirichlet L-function, $L(s, \chi)$.

3 Lecture 3: Hasse-Weil L-functions

(1) The hyperplane at infinity is defined to be

$$\{[a_0, a_1, \ldots, a_n] \in \mathbb{P}^n(\mathbb{F}_{q^m}) : a_0 = 0\}$$

(a) Find the number of points on this hyperplane for any m, n and prime power q.
(b) Using the answer from part (a), find the local zeta function of this hyperplane.

(2) Prove that the product defining $L_E(s)$ converges and is analytic for all $\mathrm{Re}(s) > 3/2$.
Hint: use Hasse's Theorem that $|a_{E,q}| \leq 2\sqrt{q}$.

(3) Let $E : Y^2 = X^3 - X$.
(a) Using Pari, etc, calculate directly a_p for $p \leq 17$, p, prime.
Recall that $a_{p^e} = p^e + 1 - |E(\mathbb{F}_{p^e})|$, where we are counting all the points in $\mathbb{P}^2$.
(b) Calculate a_n for all $n \leq 17$, using the facts that
(i) $a_{mn} = a_m a_n$ if $\gcd(m, n) = 1$ and (ii) $a_{p^e} = a_p a_{p^{e-1}} - p a_{p^{e-2}}$.
(c) Compare your results from part (b) with the data in the LMFDB:
`http://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/32/2/1/a/`
(d) Calculate a_9 directly by counting solutions.
If your result differs from the value you obtained in part (b), try to explain why.

4 Lecture 4: The Big Picture. . . we believe

(1) Suppose that $F \in \mathcal{S}$ and let θ be a fixed real number.
Show that if $F(s)$ is regular at $s = 1$, then $F(s + i\theta)$ is also an element of $\mathcal{S}$.

(2) It was stated in the lectures that the Dirichlet L-function for an imprimitive character is not in the Selberg class.
What conditions in the definition of the Selberg class fail for such L-functions?

(3) Show that Dedekind zeta-functions are elements of the Selberg class.

(4) Prove that the Selberg class is multiplicative.