ALGEBRAIC NUMBER THEORY – PROBLEM SET 3

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(1) Prove that if K is a number field, $x \in K$ is a unit if and only if $x \in \mathcal{O}_K$ and $N(x) = \pm 1$.

(2) (a) Show that the polynomial $x^5 - x + 1$ is irreducible over $\mathbb{Q}$ (Hint: reduce mod 5). Let α be one of its roots. Calculate the integers r_1 and r_2 for the field $\mathbb{Q}[\alpha]$.
   (b) Calculate the discriminant of $(1, \alpha, \ldots, \alpha^4)$. Show that it is square free and deduce that $\mathbb{Z}[\alpha]$ is the ring of integers of $\mathbb{Q}[\alpha]$.
   (c) Show that $\mathbb{Z}[\alpha]$ is principal. (Hint: Reduce $x^5 - x + 1$ modulo 2 and 3 and show that $\mathbb{Z}[\alpha]$ contains no ideals of norm 2 pr 3.)

(3) The rings $\mathbb{Z}[\sqrt{6}]$ and $\mathbb{Z}[\sqrt{7}]$ are PIDs. Exhibit generators for the ideals $(3, \sqrt{6}), (5, 4+\sqrt{6}), (2, 1+\sqrt{7})$. (Hint: Compute the norm of each of the given ideals of the form (p, α) and find an element $\beta \in \mathcal{O}_K$ of suitable norm.)

(4) Find the prime factorizations of the ideals $(3), (5)$ and (7) in $\mathbb{Z}[\sqrt{-5}]$. Show that the prime ideal factors in (7) are not principal.

(5) Let $K = \mathbb{Q}[\sqrt[3]{5}]$. Given that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{5}]$, find the prime factorization of the ideals $(2), (3), (5)$, and (7) in $\mathcal{O}_K$. Show that all the prime ideal factors which occur are principal. Using Minkowski's bound, deduce that $\mathcal{O}_K$ is a PID.

(6) Calculate the class number for $K = \mathbb{Q}[\sqrt{-23}]$.

(7) Find a fundamental unit for the field $\mathbb{Q}[\sqrt{67}]$.

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