ALGEBRAIC NUMBER THEORY – PROBLEM SET 2

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(1) In the previous problem set, you showed that each of the following numbers is algebraic:
$$\frac{1}{2}, \sqrt{-5}, \sqrt{17}+\sqrt{19}, e^{2\pi i/7}.$$
  (a) Assuming that the polynomials you found are irreducible, what are the conjugates of these numbers over $\mathbb{Q}$?
  (b) Calculate their trace and norm over $\mathbb{Q}$.

(2) Let $K = \mathbb{Q}[\sqrt{d}]$ where d is a square free integer. Describe the embeddings σ_1, σ_2 of K into $\mathbb{C}$. Are the fields $\sigma_1(K)$ and $\sigma_2(K)$ different?

(3) Let $K = \mathbb{Q}(\alpha)$ where $\alpha^3 = m$ and m is not a cube. Evaluate $D(1, \alpha, \alpha^2)$ by the formula $\det(\sigma_i(x_j)^2)$. Write down the traces of $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence evaluate $D(1, \alpha, \alpha^2)$ by the formula involving traces. Which do you prefer?

(4) Suppose that β is a root of $X^3 + pX + q = 0$, where $X^3 + pX + q$ is an irreducible polynomial in $\mathbb{Z}[X]$. Verify that $1, \beta, \beta^2, \beta^3$ have traces $3, 0, -2p, -3q$, respectively, and compute $\mathrm{Tr}(\beta^4)$. Deduce that $D(1, \beta, \beta^2) = -4p^3 - 27q^2$. Use similar ideas to deduce the classical discriminant formula for quadratic polynomials.

(5) (a) Let B be a free $\mathbb{Z}$-module of rank n, and let N be a submodule of finite index generated by elements $\gamma_1\gamma_2, \ldots, \gamma_n$ of B. Then
$$D(\gamma_1, \ldots, \gamma_n) = [B : N]^2 * disc(B/\mathbb{Z}).$$
  (b) Let α be a root of $X^3 - X - 1$, which is irreducible over $\mathbb{Q}$ (check). Show that $\{1, \alpha, \alpha^2\}$ is a $\mathbb{Z}$-basis for $\mathbb{Q}[\alpha]$ (hint: compute the discriminant and use part (a)).

(6) Let $K = \mathbb{Q}[\alpha]$ with $\alpha^3 = 2$. Show that $\mathcal{O}_K$ is principal. (Hint: finiteness of class number.)

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