ALGEBRAIC NUMBER THEORY – PROBLEM SET 1

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(1) Show that each of the following numbers is algebraic:
$$\frac{1}{2}, \sqrt{-5}, \sqrt{17}+\sqrt{19}, e^{2\pi i/7}.$$

(2) (Example from class)

Definition 0.0.1. Any extension field of degree 2 over the field $\mathbb{Q}$ of rational numbers is called a *quadratic field*.

(a) Show that every quadratic field is of the form $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer.
(b) Let $\alpha \in K = \mathbb{Q}(\sqrt{d})$. Assume $\alpha = a + b\sqrt{d}, b \neq 0, a, b \in \mathbb{Q}$. Show that α is an algebraic integer if and only if $2a \in \mathbb{Z}$ and $a^2 - bd^2 \in \mathbb{Z}$. (Hint: what is the minimal polynomial of α?)
(c) Show that if $d \equiv 2, 3 \bmod 4$, the ring A of integers of K consists of all elements of the form $a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$.
(d) Show that if $d \equiv 1 \bmod 4$, A consists of all elements of the form $\frac{1}{2}(u+v\sqrt{d})$ with $u, v \in \mathbb{Z}$ of the same parity.

(3) (a) Give an example of an integral domain A that is not integrally closed.
(b) Since A is not integrally closed, then it's not a unique factorization domain (why?). Give an example of an element of A that has two distinct factorizations into irreducible elements.

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