Exercises for algebraic curves

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1 Exercise Lecture 1

1.1 Exercise

Show that $V = \{(x, y) \in \mathbb{C}^2 \text{ s.t. } y = \sin x\}$ is not an algebraic set.

1.2 Exercise

If V is a algebraic set and $P \notin V$ a point, show that there exists a polynomial F such that $F(x) = 0$ for all $x \in V$ and $F(P) = 1$.

1.3 Exercise

1. Let $F(x, y, z)$ be a homogeneous polynomial of degree d over a field k. Show (Euler relation)

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = d \cdot F(x, y, z)$$

   (hint: take partial derivative of $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$ with respect to λ and then $\lambda = 1$).

2. If d is coprime to the characteristic of k, show that the projective set $C : F = 0$ is singular at $P_0 = (x_0 : y_0 : z_0)$ if and only if

$$\left(\frac{\partial F}{\partial x}(P_0), \frac{\partial F}{\partial y}(P_0), \frac{\partial F}{\partial z}(P_0)\right) = (0, 0, 0).$$

3. If P_0 is not singular, an equation of the tangent at P is

$$\frac{\partial F}{\partial x}(P_0)x + \frac{\partial F}{\partial y}(P_0)y + \frac{\partial F}{\partial z}(P_0)z = 0.$$

4. Study the singularities of $C_1/\mathbb{C} : y^2 z = x^3$ over $\mathbb{C}$..

1.4 Exercise

Let V be the projective variety defined by $Y^2Z - (X^3 + Z^3) = 0$. Show that the map $\phi : V \to \mathbb{P}^2$ given by $(X : Y : Z) \mapsto (X^2 : XY : Z^2)$ is a morphism.

1.5 Exercise

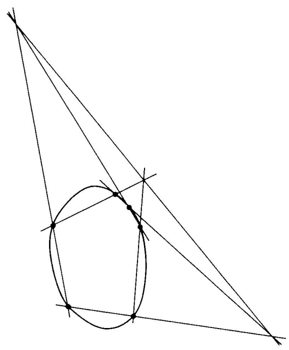
Show the following result:

If two projective plane curves C_1, C_2 of degree n intersect in exactly n^2 points and that there exists a irreducible curve D of degree $m < n$ containing mn of these points, then there exists a curve of degree at most $n - m$ containing the $n(n - m)$ residual points.

To do so, let F_1, F_2, G the equations of C_1, C_2 and D and $p = [a : b : c]$ be a point of D which is not in $C_1 \cap C_2$. Show that there exists a linear combination of F_1 and F_2 containing p. Conclude using Bézout.

One can use this to prove the following corollary (Pascal mystical hexagon): The opposite sides of a hexagon inside an irreducible conic meet in three collinear points.

Figure 1: Pascal mystical hexagon



2 Exercise Lecture 2

2.1 Exercise

Let $C = V(F) \subset \mathbb{P}^2$ be a dimension 1 affine variety over k. Let $P \in C$ be a smooth point. We are going to show that $\overline{k}[C]_P$ is a discrete valuation ring and that if $L = V(aX + bY + c)$ is any line through P which is not tangent to C at P, then its image in $\overline{k}[C]_P$ is a uniformizer at P

1. Show that by a change of variables we can assume that $P = (0, 0)$ that $Y = 0$ is the tangent at P and that $L = V(X)$.

2. Show that $\mathcal{M}_P = (X, Y)$

3. Show that $F = YG - X^2H$ where $G = a +$ higher terms with $a \neq 0$ and $H \in \overline{k}[X]$.

4. Conclude that $\mathcal{M}_P = (X)$.

2.2 Exercise

Let $\phi : C_1 \to C_2$ be a non-constant morphism of curves and $f \in \overline{k}(C_2)^*, P \in C_1$. Prove that

$$\operatorname{ord}_P(\phi^* f) = e_\phi(P) \operatorname{ord}_{\phi(P)}(f).$$

2.3 Exercise

We give a proof of residue theorem in the case of $C = \mathbb{P}^1$ over an algebraically closed field k.

1. Consider a rational fraction $P(X)/Q(X)$. Show that one can write P/Q as a sum of terms of the form $c(X - a)^n$ with $c \in k^*, a \in k$ and $n \in \mathbb{Z}$. By linearity, one can restrict to one of these cases.

2. Show for each cases that the formula holds.

3 Exercise Lecture 3

3.1 Exercise

Prove that a curve C has genus 0 iff there exists two distinct points $P, Q \in C$ such that $(P) \sim (Q)$.

3.2 Exercise

Let $\phi : C_1 \to C_2$ a non-constant morphism between curves.

1. Show that $g_{C_1} \geq g_{C_2}$.

2. Prove that if there is equality then $g = 0$ or ($g = 1$ and ϕ is unramified) or ($g \geq 2$ and ϕ is an isomorphism).

3.3 Exercise

Let k be an algebraically closed field. Let C be a curve of genus $g_C > 1$ and G be the group of automorphisms of C. It is known that this is always a finite group. In the first part of this exercise, we are going to prove this result when C is hyperelliptic and the characteristic of k is different from 2.

We write $C : Y^2 = f(X)$ where f is of degree $2g_C + 2$ (a singular model for C). Recall that isomorphisms of hyperelliptic curves are of the form

$$g : (X, Y) \mapsto \left(\frac{aX + b}{cX + d}, \frac{eY}{(cX + d)^{g+1}} \right)$$

with $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(k)$ and $e \in k^*$. We denote $\widetilde{g}$ the induces automorphism of $\mathbb{P}^1$ given by $(X : Z) \mapsto (aX + bZ : cX + dZ)$ and we therefore have a surjective morphism from G to $\widetilde{G} = \{\widetilde{g},\ g \in G\}$.

1. Show that the kernel of this morphism is generated by the hyperelliptic involution ι.

Hence in order to prove that G is finite, it is enough to prove that $\widetilde{G}$ is. Let $\widetilde{g} \in \widetilde{G}$.

2. Show that the $2g+2$ points $(x_i, 0) \in C$ where x_i are the roots of f are the fixed points of ι.

3. Show that $\widetilde{g}$ permutes the points $Q_i = (x_i : 1)$.

4. Show that an automorphism of $\mathbb{P}^1$ which fixes 3 distinct points is the identity.

5. Conclude that there exists an injective morphism from $\widetilde{G}$ into Sym_{2g+2} and that $\#G \leq 2(2g + 2)!$.

6. Describe briefly how to compute the elements of G given a factorization of f.

We now come back to the case where C is not necessarily hyperelliptic and we assume that G is finite. We assume also that the characteristic of k does not divide $\#G = n$.

We know that there exists a curve D/k and a morphism $\phi : C \to D$ separable of degree n such that for all $Q \in D$, $\phi^{-1}(Q) = \{g(P),\ g \in G\}$, where $P \in C$ is any point such that $\phi(P) = Q$ (the curve D is the "quotient" of C by G and in particular $\phi \circ g = \phi$ for all $g \in G$). Let $P \in C$ be a point with ramification index $e_\phi(P) = r$.

7. Show that $\phi^{-1}(\phi(P))$ consists of exactly n/r points, each of ramification index r.

Let $P_1, \dots, P_s$ be a maximal set of ramification points of C lying over distinct points of D and let $e_\phi(P_i) = r_i$.

8. Show that Riemann-Hurwitz formula implies

$$\frac{2g_C - 2}{n} = 2g_D - 2 + \sum_{i=1}^{s} 1 - \frac{1}{r_i}.$$

9. As $g_C \geq 2$, then the left side is > 0. Show that if $g_D \geq 0$, $s \geq 0$, $r_i \geq 0$ are integers such that

$$2g_D - 2 + \sum_{i=1}^{s} 1 - \frac{1}{r_i} > 0$$

then the minimal value of this expression is $1/42$.

10. Conclude that $n \leq 84(g_C - 1)$.