The modular group

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Image by Arnaud Chéritat

Isomorphism classes of complex tori

We have seen in the first lecture that to any lattice $\Lambda \subset \mathbb{C}$ we can associate a complex tori defined as $\mathbb{C}/\Lambda$.

Question

When two distinct lattice Λ_1 and Λ_2 give rise to isomorphic complex tori $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$?

Definition

Two lattice Λ_1 and Λ_2 are said to be **homothetic** if there exists $\alpha \in \mathbb{C}$ such that $\alpha\Lambda_1 = \Lambda_2$

If α is such that $\alpha\Lambda_1 \subseteq \Lambda_2$, then we can define a surjective map $\phi_\alpha : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$, by setting

$$\phi_\alpha([z]_{\Lambda_1}) = [\alpha z]_{\Lambda_2}.$$

It can be shown that this is a holomorphic map. Moreover if $\alpha\Lambda_1 = \Lambda_2$ then ϕ_α is an isomorphism.

Isomorphism classes of complex tori

Consider the association

$$\{\alpha\in\mathbb{C} : \alpha\Lambda_1 \subseteq \Lambda_2\} \to \left\{ \frac{\mathbb{C}}{\Lambda_1} \xrightarrow{\phi} \frac{\mathbb{C}}{\Lambda_2} : \phi(0) = 0 \text{ and } \phi \text{ holomorphic} \right\}$$
$$\alpha \mapsto \phi_\alpha$$

Theorem

Let Λ_1 and Λ_2 be two lattices. Then the above association is a bijection. Moreover $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic if and only if Λ_1 and Λ_2 are homothetic.

Let

$$\mathcal{L} = \{\Lambda \subset \mathbb{C} : \Lambda \text{ is a lattice}\}$$

We have an action of $\mathbb{C}^*$ on $\mathcal{L}$

$$\begin{aligned} \mathbb{C}^* \times \mathcal{L} &\longrightarrow \mathcal{L} \\ (\alpha, \Lambda) &\longmapsto \alpha\Lambda \end{aligned}$$

Let $\Lambda_0 \in \mathcal{L}$ the **orbit** of Λ_0 under $\mathbb{C}^*$ is the set

$$\mathrm{Orb}_{\mathbb{C}^*}(\Lambda_0) = \{\Lambda \in \mathcal{L} : \Lambda = \alpha\Lambda_0\}$$

The set of orbits is usually denoted by $\mathcal{L}/\mathbb{C}^*$. Since two complex tori are isomorphic if and only if the associates lattices are homothetic we have that there exists a bijection

$$\mathcal{L}/\mathbb{C}^* \longrightarrow \{\text{isomorphisms classes of complex tori}\}$$

So we want to understand better what $\mathcal{L}/\mathbb{C}^*$ looks like.

Lemma

Every lattice is homothetic to a lattice of the form $\mathbb{Z} + \tau\mathbb{Z}$ *with* $\mathrm{Im}(\tau) > 0$*.*

Let

$$\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

denote the upper half plane. Then, as consequence of the previous lemma, we have that the map

$$\begin{aligned} \mathfrak{h} &\longrightarrow \mathcal{L}/\mathbb{C}^* \\ \tau &\longmapsto [\Lambda_\tau] \end{aligned}$$

is surjective. Unfortunately is not injective.

We have to consider another action to actually get a set of representative for the set of orbits. First of all recall that $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ acts on $\mathfrak{h}$ by fractional linear transformation, which means that given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we let

$$\gamma \cdot z = \frac{az+b}{cz+d}$$

Lemma

If $\mathrm{Im}(\tau) \neq 0$, *then*

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)}{|cz+d|^2}\,\mathrm{Im}(z)$$

Let

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) : ad - bc = 1 \right\}.$$

$\mathrm{SL}_2(\mathbb{Z})$ is called the modular group and often denote by $\Gamma(1)$. By the above lemma $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{C}$ by fractional linear transformation:

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{h} \longrightarrow \mathfrak{h}$$
$$\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \longmapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

Clearly $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts like the identity. Thus we get an action of the quotient group $\overline{\Gamma(1)}$ on $\mathfrak{h}$

Lemma

The lattices Λ_{τ_1} *and* Λ_{τ_2}, $(\tau_1, \tau_2 \in \mathfrak{h})$, *are homothetic if and only if there exist* $\gamma \in \Gamma(1)$ *such that* $\gamma(\tau_1) = \tau_2$.

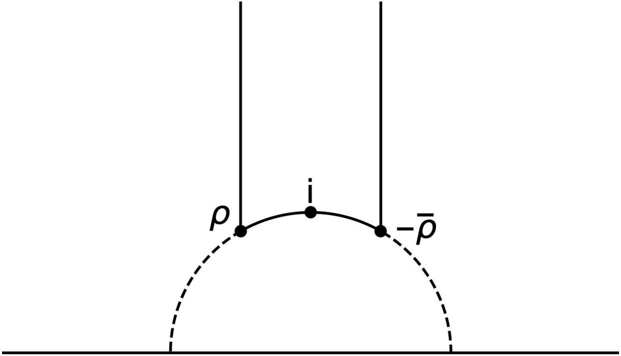
Thus to understand $\mathcal{L}/\mathbb{C}^*$ we only need to compute a fundamental domain for the action of $\Gamma(1)$ on $\mathfrak{h}$

Recall that $\mathcal{F}$ is a fundamental domain for the action of the modular group $\Gamma(1)$ if the following condition holds: each orbit intersect $\mathcal{F}$ in exactly one point.

A fundamental domain

Let

$$\tilde{\mathcal{F}} = \left\{\tau \in \mathfrak{h} : |\tau| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}\right\}$$



where $\rho = \exp(\frac{2\pi i}{3})$

Matrices in $\Gamma(1)$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and their action is given by

$$T(\tau) = \tau + 1 \qquad S(\tau) = -\frac{1}{\tau}$$

Note that $S^2 = I$. Moreover

$$(ST)^3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

and

$$(TS)^3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

So $(TS)^3 = I$ and $(ST)^3 = I$ in $\overline{\Gamma(1)}$.

Theorem

(1) *Given* $\tau \in \mathfrak{h}$ *then there exists* $\gamma \in \Gamma(1)$ *such that* $\gamma(\tau) \in \tilde{\mathcal{F}}$

(2) *Given* $\tau \in \tilde{\mathcal{F}}$ *and* $\gamma \in \Gamma(1)$*, then* $\gamma(\tau) \in \tilde{\mathcal{F}}$ *if and only if one of the following situations occur:*

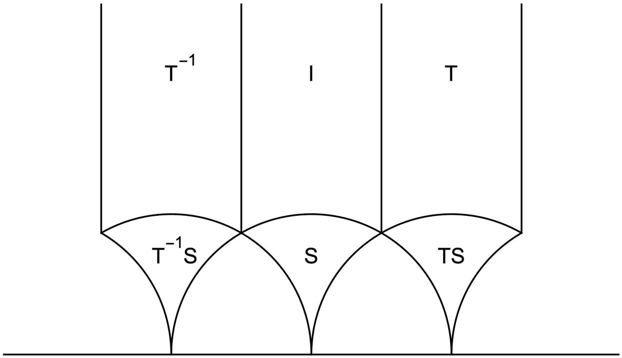
   (a) $\mathrm{Re}(\tau) = \frac{1}{2}$ *and* $\gamma(\tau) = \tau - 1$
   (b) $\mathrm{Re}(\tau) = -\frac{1}{2}$ *and* $\gamma(\tau) = \tau + 1$
   (c) $|\tau| = 1$ *and* $\gamma(\tau) = -\frac{1}{\tau}$

(3) *Given* $\tau \in \mathfrak{h}$ *let*

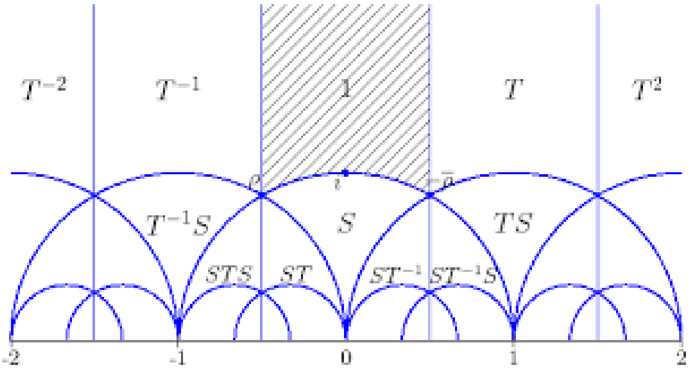
$$\mathrm{Stab}_\tau = \left\{ \gamma \in \overline{\Gamma(1)} : \gamma(\tau) = \tau \right\}.$$

Then for $\tau \in \tilde{\mathcal{F}}$ *we have*

$$\mathrm{Stab}_\tau = \begin{cases} < S > & \text{if } \tau = i \\ < ST > & \text{if } \tau = \rho = \exp(\frac{2\pi i}{3}) \\ < TS > & \text{if } \tau = -\bar{\rho} = \exp(\frac{\pi i}{3}) \\ I & \text{otherwise.} \end{cases}$$



T^{-1}
I
T
$T^{-1}S$
S
TS


T^{-2}
T^{-1}
1
T
T^{2}
ρ
i
$-\bar{\rho}$
$T^{-1}S$
S
TS
STS
ST
ST^{-1}
$ST^{-1}S$
-2
-1
0
1
2

Theorem

The group $\overline{\Gamma(1)}$ is generated by T and S.

The fondamental domain for the action of $\overline{\Gamma(1)}$ on $\mathfrak{h}$.