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Pieter Moree

Forbidden integer ratios of consecutive power sums

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1 Introduction

This is a report of the results obtained in joint work Pieter Moree (Bonn) and Ioulia Baoulina (Moscow), starting by providing background. For the details see [1].

For natural numbers $m, k \geqslant 1$ we consider the power sum

$$S_k(m) = 1^k + 2^k + \cdots + (m-1)^k.$$

For $k = 1, 2, 3$, $S_k(m)$ equals, respectively,

$$\frac{m(m-1)}{2}, \frac{(m-1)m(2m-1)}{6}, \frac{m^2(m-1)^2}{4}.$$

In the 17th century J. Faulhaber (1580-1635) realized that the power sums can be, in essence, expressed as polynomials in $S_1(m)$. Namely, there exists polynomials F_k and G_k such that

$$S_k(m) = \begin{cases} F_k(S_1(m)) \text{ with } \deg(F_k) = (k+1)/2 \text{ if } k \text{ is odd;} \\ S_2(m)G_k(S_1(m)) \text{ with } \deg(G_k) = (k-2)/2 \text{ if } k \text{ is even.} \end{cases}$$

The following theorem expresses the power sum $S_k(m)$ in terms of Bernoulli numbers B_k, which are defined by the identity

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Theorem 1 (Faulhaber) *For all positive integers m and k, we have*

$$S_k(m) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j m^{k+1-j}.$$

E. Kummer in 1850 gave the following definition of an irregular prime.

Definition 2 *Write $B_k = \frac{u_k}{v_k}$ with $(u_k, v_k) = 1$. An odd prime p is called **irregular** if $p \mid u_k$ for some $k \in \{2, 4, \ldots, p-3\}$, and the pair (k, p) is called an **irregular pair**. An odd prime is called **regular** if it is not irregular.*

In 1851, Kummer obtained the following congruence, which plays an important role in the development of the theory of p-adic zeta functions.

Theorem 3 (Kummer) *If $\ell \equiv k \not\equiv 0 \pmod{p-1}$, then*

$$\frac{B_\ell}{\ell} \equiv \frac{B_k}{k} \pmod{p}.$$

Furthermore, he proved Fermat's Last Theorem for regular prime exponents.

Theorem 4 (Kummer) *If p is regular, then $x^p + y^p = z^p$ has only trivial solutions.*

In his work on Fermat's Last Theorem, Kummer also showed that p is regular when the class number $h_p = h(\mathbb{Q}(\zeta_p))$ of the pth cyclotomic field is not divisible by p.

Conjecture 5 (Kellner, 2011) *[3] Let m and k be positive integers with $m \geqslant 3$. Then the ratio*

$$\frac{S_k(m+1)}{S_k(m)} \text{ is an integer if and only if } (m,k) \in \{(3,1),(3,3)\}.$$

Hence, since $S_k(m+1) = S_k(m) + m^k$, we have

$$\frac{S_k(m+1)}{S_k(m)} \in \mathbb{Z} \quad \textit{iff} \quad \frac{m^k}{S_k(m)} \in \mathbb{Z}.$$

Kellner's conjecture is thus equivalent with the following one (in a moment we will see what Erdős and Moser have to do with it).

Conjecture 6 (Kellner–Erdős–Moser) *Let a, k, m be positive integers with $m \geqslant 3$. Then*

$$aS_k(m) = m^k \iff (a,k,m) \in \{(1,1,3),(3,3,3)\}.$$

In case $m = 3$ we have $aS_k(3) = 3^k$ and it follows that $a = 3^e$ for some $e \geq 0$. Then $1 + 2^k = 3^{k-e}$, which has as only solutions $1 + 2 = 3$ and $1 + 2^3 = 3^2$ (as was already known in the Middle Ages).

In case $a = 1$, we obtain the following special case of the Kellner-Erdős-Moser conjecture.

Conjecture 7 (Erdős, 1950) *The Diophantine equation*

$$1^k + 2^k + \cdots + (m-1)^k = m^k \tag{1}$$

has only one solution, namely $1 + 2 = 3$.

A few years after Erdős made his conjecture L. Moser proved the following theorem.

Theorem 8 (Moser, 1953) *[7], cf. [4] If (m,k) is a solution of (1) with $k \geqslant 2$, then $m > 10^{10^6}$.*

The lower bound for m can be sharpened to $m > 10^{9 \cdot 10^6}$, see P. Moree [4]. In 2011, Y. Gallot, P. Moree and W. Zudilin [2] using completely different methods again sharpened the lower bound.

Theorem 9 *[2] If (m, k) is a solution of (1) with $k \geqslant 2$, then $m > 10^{10^9}$.*

For the general case $aS_k(m) = m^k$, in 2015, I. Baoulina and P. Moree [1] established the following results.

Theorem 10 *If $aS_k(m) = m^k$ with $m > 3$, then*

- *a has no regular prime divisors;*
- *$a = 1$ or $a > 1500$;*
- *m has no regular prime divisors;*
- *$k, m > 10^{82}$;*
- *$k, m > 10^{9 \cdot 10^6}$ if $m \equiv 1 \pmod 3$;*
- *$k, m > 10^{4 \cdot 10^{20}}$ if $m \equiv 1 \pmod{30}$.*

Theorem 11 *Suppose that (m, k) is a non-trivial solution of $aS_k(m) = m^k$ and p is a prime dividing m. Then*

- *p is an irregular prime;*
- *$p^2 \mid u_k$;*
- *$k \equiv r \pmod{p-1}$ for some irregular pair (r, p).*

In case $a = 1$ this result is due to P. Moree, H. te Riele and J. Urbanowicz [6].

Corollary 12 *If a has a regular prime divisor, then the equation*

$$aS_k(m) = m^k$$

has only trivial solutions.

In 1915, K. L. Jensen proved the following theorem.

Theorem 13 *There are infinitely many primes* $p \equiv 5 \pmod{6}$ *that are irregular.*

Note that it is still not known whether there are infinitely many regular primes. Let us define

$$\pi_i(x) := \#\{p \leqslant x : p \text{ is irregular}\}.$$

In 1954, C. L. Siegel provided an heuristic argument to justify the conjecture that

$$\pi_i(x) \sim \left(1 - \frac{1}{\sqrt{e}}\right)\pi(x) \sim 0.39\ldots\frac{x}{\log x}.$$

We will make the following weaker conjecture.

Conjecture 14 *There exists* $\delta \in (0, 1)$ *such that*

$$\pi_i(x) < (1 - \delta)\frac{x}{\log x} \text{ as } x \to \infty.$$

Let I be the set of integers composed solely of irregular primes. Suppose that conjecture (14) holds true. The standard theory of the average behaviour of arithmetical functions yields that $I(x) \ll x(\log x)^{-\delta}$. On combining this estimate and Corollary 12 we then obtain the following result.

Proposition 15 *Under Conjecture 14 the set of integer ratios that are of the form* $S_k(m + 1)/S_k(m)$ *with* $m \geq 3$ *has zero natural density.*

We now briefly consider how to deal with $aS_k(m) = m^k$ for a prescribed a.

A pair $(t, q)_a$ with q a prime and $2 \leqslant t \leqslant q - 3$ even is called **helpful** if $q \nmid a$ and, for every $c = 1, 2, \ldots, q - 1$, we have

$$aS_t(c) \not\equiv c^t \pmod{q}.$$

If q is an irregular prime, we require in addition that (t, q) should not be an irregular pair.

Lemma 16 *[1] If $(t,q)_a$ is a helpful pair and (m,k) a solution of*

$$aS_k(m) = m^k$$

with k even, then $k \not\equiv t \pmod{q-1}$.

Suppose that $1 < a \leqslant 1500$. Then the equation $aS_k(m) = m^k$ has no non-trivial solutions except possibly when a is an irregular prime or $a = 37 \times 37$. We have $\pi(1500) = 239$, $\pi_i(1500) = 90$ and $\frac{90}{239} \approx 0.38$.

Example 17 *Consider $673S_k(m) = m^k$; $(408, 673)$, $(502, 673)$ are the irregular pairs.*
Reduction modulo 5:

- $3S_k(m) \equiv m^k \pmod 5$
- $k \equiv 502 \pmod{672} \subset k \equiv 2 \pmod 4$
- $(2,5)_3$ *is helpful*

Reduction modulo 17:

- $10S_k(m) \equiv m^k \pmod{17}$
- $k \equiv 408 \pmod{672} \subset k \equiv 8 \pmod{16}$
- $(8,17)_{10}$ *is helpful*

So, the equation has no solutions.

2 Start of Moser's proof of Theorem 8

Consider a prime p so that m^k takes a simple form modulo p. The most obvious choice is to take p to be a prime divisor of $m-1$. On using that the power sum as a function of k is periodic modulo p, the equation (1) reduces to

$$S_k(m) \equiv \frac{m-1}{p}\left(1^k + 2^k + \cdots + (p-1)^k\right) \equiv m^k \equiv 1 \pmod p. \quad (2)$$

Proposition 18 *[4] Let $p \mid m-1$ be a prime. Modulo p we have*

$$S_k(p) \equiv \begin{cases} -1 & \text{if } p-1 \text{ divides } k; \\ 0 & \text{otherwise.} \end{cases}$$

By the proposition we have $S_k(p) \equiv -1 \pmod{p}$, and hence by (2) we must have

$$\frac{m-1}{p} + 1 \equiv 0 \pmod{p}.$$

We conclude that $m-1$ must be squarefree and hence that

$$\prod_{p \mid m-1} \left(\frac{m-1}{p} + 1 \right) \equiv 0 \pmod{m-1},$$

On expanding the product we obtain

$$\prod_{p \mid m-1} \left(\frac{m-1}{p} + 1 \right) = 1 + \sum_{p \mid m-1} \frac{m-1}{p} + \sum_{\substack{p_1, p_2 \mid m-1 \\ p_1 \neq p_2}} \frac{(m-1)^2}{p_1 p_2} + \cdots,$$

where the sum involving the primes p_1, p_2 and the sums not indicated involving three primes or more are divisible by $m-1$. Hence we obtain

$$\sum_{p \mid m-1} \frac{m-1}{p} + 1 \equiv 0 \pmod{m-1},$$

which on division by $m-1$ gives

$$\sum_{p \mid m-1} \frac{1}{p} + \frac{1}{m-1} \in \mathbb{Z}_{\geqslant 1}. \tag{3}$$

Writing the equation $S_k(m) = m^k$ as $S_k(m+2) = 2m^k + (m+1)^k$ and using the proposition, we get

$$\sum_{p \mid m+1} \frac{1}{p} + \frac{2}{m+1} \in \mathbb{Z}_{\geqslant 1}. \tag{4}$$

By similar ad hoc arguments one is led to the following two conclusions:

$$\sum_{p|2m-1} \frac{1}{p} + \frac{2}{2m-1} \in \mathbb{Z}_{\geqslant 1}; \tag{5}$$

$$\sum_{p|2m+1} \frac{1}{p} + \frac{4}{2m+1} \in \mathbb{Z}_{\geqslant 1}. \tag{6}$$

On adding the four equations (3), (4), (5) and (6), we obtain

$$\sum_{p|M} \frac{1}{p} + \frac{1}{m-1} + \frac{2}{m+1} + \frac{2}{2m-1} + \frac{4}{2m+1} \geqslant 3\frac{1}{6},$$

where $M = (m^2-1)(4m^2-1)/12$. Using the fact that $\sum_{p\leqslant 10^7} \frac{1}{p} < 3.16$, we find $M > \prod_{p\leqslant 10^7} p$. This gives $m > 10^{10^6}$.

Details of the proof can be found in P. Moree [4] and L. Moser [7]. The title of [4] refers to the four, in an ad hoc way derived, equations (3), (4), (5) and (6) ("the four mathemagical rabbits") and the fact that they can be actually obtained from one theorem ("the top hat").

For a survey of work on the Erdős-Moser equation the reader can consult [5].

3 Challenges

- Can one use that $p^2 \mid u_k$ (with $p \mid a$), rather than $p \mid u_k$?
- Show that Conjecture 7 implies Conjecture 6.
- Write a program to deal with $aS_k(m) = m^k$ for a given a.
- Show that if $S_k(m) = bm^k$, then $120 \mid k$.
- Study the equation $aS_k(m) = bm^k$.

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