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New results on the Class Number One problem for Functions Fields

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Let p be a prime number and let q be a power of p. This talk was devoted to the classification of algebraic function fields in one variable over the finite field $\mathbb{F}_q$ with class number one. An immediate consequence of the Riemann-Roch theorem is that every algebraic function field in one variable over $\mathbb{F}_q$ with genus 0 has class number 1. In 1971 Mac Rae classified algebraic function fields in one variable over $\mathbb{F}_q$ with positive genus and class number 1 having rational places. In 1972 Madan and Queen gave a full list of zeta functions of the algebraic function fields in one variable over $\mathbb{F}_q$ with positive genus and class number 1. They also proved that an algebraic function field in one variable over $\mathbb{F}_q$ with genus greater than 4, has class number greater than 1.

The following theorem was the first attempt to give a complete classification:

Theorem 1 (Leitzel, Madan, Queen – 1975) *Let K be an algebraic function field in one variable over $\mathbb{F}_q$ with genus g such that $0 < g < 4$ and class number 1. Then K is isomorphic to the algebraic function field $\mathbb{F}_q(x, y)$ in one variable defined by one of the following equations:*

(i) $y^2 + y + x^3 + x + 1 = 0$, *with* $q = 2$ *and* $g = 1$*;*

(ii) $y^2 + y + x^5 + x^3 + 1 = 0$, *with* $q = g = 2$*;*

(iii) $y^2 + y + (x^3 + x^2 + 1)(x^3 + x + 1)^{-1} = 0$, *with* $q = g = 2$*;*

(iv) $y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + x^4 + x + 1 = 0$, *with* $q = 2$ *and* $g = 3$*;*

(v) $y^4 + (x^3 + x + 1)y + x^4 + x + 1 = 0$, *with* $q = 2$ *and* $g = 3$*;*

(vi) $y^2 + 2x^3 + x + 1 = 0$, *with* $q = 3$ *and* $g = 1$*;*

(vii) $y^2 + y - x^3 + \alpha = 0$, *with* $q = 4$, $\alpha \in \mathbb{F}_4^\times$ *and* $g = 1$, *where* α *is a generator of the multiplicative group* $\mathbb{F}_4^\times$.

In what follows, an algebraic function field in one variable over $\mathbb{F}_2$ with genus 4 and class number 1 is constructed and this leads to a complete classification of the algebraic function fields in one variable over $\mathbb{F}_q$ with class number one.

Definition 1 *An algebraic curve defined over* $\mathbb{F}_q$ *is called a* n-pointless curve *if it has no* $\mathbb{F}_{q^m}$ *rational points for each* $m \leq n$*. Similarly, an algebraic function field in one variable over* $\mathbb{F}_q$ *corresponding to a* n*-pointless curve is called a* n-pointless function field.

Every genus 0 algebraic curve defined over $\mathbb{F}_q$ has $\mathbb{F}_q$-rational points (this follows from the Riemann hypothesis for function fields, proved by Weil in 1948). Also, in 1936, Hasse proved that each genus 1 algebraic curve defined over $\mathbb{F}_q$ has $\mathbb{F}_q$-rational points.

In 2013 (cf [8]) Stirpe proved that, for any positive integer n, there exists an algebraic function field in one variable over $\mathbb{F}_q$ without places of degree smaller than n with genus smaller than Cq^n, where $C > 0$ is a suitable constant depending only on the prime p. This construction, with $n = 3$, gives us the algebraic function field $\mathbb{F}_2(x, y)$ in one variable defined by the following equation:

$$\begin{aligned} & y^5 + y^3 + y^2(x^3 + x^2 + x)+ \\ & \quad + y\frac{x^7 + x^5 + x^4 + x^3 + x}{x^4 + x + 1}+ \\ & \quad + \frac{x^{13} + x^{12} + x^8 + x^6 + x^2 + x + 1}{(x^4 + x + 1)^2} = 0. \end{aligned}$$

This algebraic function field over $\mathbb{F}_2$ is 3-pointless, has genus 4 and class number 1. From now on, we denote it by L.

Theorem 2 (Mercuri, Stirpe – 2015) *Let K be an algebraic function field in one variable over $\mathbb{F}_2$ with genus 4 and class number 1. Then K is isomorphic to L.*

Using this result, the classification is given in the following way:

Theorem 3 (Leitzel, Madan, Queen; Mercuri, Stirpe et al.) *Let K be an algebraic function field in one variable over $\mathbb{F}_q$ with positive genus and class number 1. Then K is isomorphic to the algebraic function field $\mathbb{F}_q(x, y)$ in one variable defined by one of the following equations:*

(i) $y^2 + y + x^3 + x + 1 = 0$, *with* $q = 2$ *and* $g = 1$;

(ii) $y^2 + y + x^5 + x^3 + 1 = 0$, *with* $q = g = 2$;

(iii) $y^2 + y + (x^3 + x^2 + 1)(x^3 + x + 1)^{-1} = 0$, *with* $q = g = 2$;

(iv) $y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + x^4 + x + 1 = 0$, *with* $q = 2$ *and* $g = 3$;

(v) $y^4 + (x^3 + x + 1)y + x^4 + x + 1 = 0$, *with* $q = 2$ *and* $g = 3$;

(vi) $y^2 + 2x^3 + x + 1 = 0$, *with* $q = 3$ *and* $g = 1$;

(vii) $y^2 + y - x^3 + \alpha = 0$, *with* $q = 4$, $\alpha \in \mathbb{F}_4^\times$ *and* $g = 1$, *where* α *is a generator of the multiplicative group* $\mathbb{F}_4^\times$*;*

(viii) $y^5 + y^3 + y^2(x^3 + x^2 + x) + y(x^7 + x^5 + x^4 + x^3 + x)(x^4 + x + 1)^{-1} + (x^{13} + x^{12} + x^8 + x^6 + x^2 + x + 1)(x^4 + x + 1)^{-2} = 0$, *with* $q = 2$ *and* $g = 4$.

At the same time, independently, Shen and Shi and also Rzedowski-Calderòn and Villa-Salvador proved the same result. The proof of Shen and Shi is a correction of the original argument of Leitzel, Madan and Queen, while Rzedowski-Calderòn and Villa-Salvador showed that there exists only one (up to isomorphism) function field with genus 4 and class number 1, without using the example found by Stirpe.

References

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