PROCEEDINGS OF THE
ROMAN NUMBER THEORY ASSOCIATION
Volume 4, *Number* 1, *March* 2019, *pages* 35-42

Evelina Viada

Rational Points on Curves

Written by Manoj Gyawali

1 Introduction

One of the oldest problem in Diophantine geometry is that of the complete determination of the set of rational (or k-rational) points, of a given algebraic curve defined over the rational numbers (or more generally over a number fields). Clearly a rational curve (i.e. a curve of genus zero) defined over a number field if it has one rational point it has infinitely many. For algebraic curves of genus one with one specified rational point (i.e. elliptic curves), we have the following, by now classical, result of Mordell and Weil.

Theorem 1.1 (Mordell-Weil Theorem) *Let E be an elliptic curve defined over a number field k. Then the set $E(k)$ of k-rational points of E is a finitely generated abelian group.*

The next case is that of algebraic curves of genus greater than 1. From now on, by a curve C we mean an algebraic curve defined over the algebraic numbers $\overline{\mathbb{Q}}$ and for k a number field. We denote the k-rational points of C by $C(k)$. Mordell conjectured in 1922 that a curve of genus at least 2 has only finitely many points over any number field. This was proven by Faltings in 1983, see [4]

Theorem 1.2 (Faltings) *Let C be a curve defined over a number field k. Suppose the genus of C is at least 2, then $C(k)$ is finite.*

Unfortunately Falting's theorem is not effective, which means in particular that there is no effective bound for the height of the points in $C(k)$. The aim of this seminar is to present an effective bound on the height of the k-rational points on some families of curves, which in turn led to the complete determination of the set of rational points for the curves of the families in question.

2 Torsion and finiteness

Let A be an abelian variety, Γ a finitely generated subgroup and X an irreducible subvariety of A. In this section we dwell briefly on the following problem: If X has a large (i.e. Zariski dense) intersection with Γ what can be said about X? It all started with the celebrated Manin-Mumford conjecture (raised independently by Manin and Mumford), proved by Raynaud [9].

Theorem 2.1 (Raynaud) *Let A be an abelian variety and Tor_A its torsion subgroup. Let $C \subset A$ be a curve of genus ≥ 2. Then, $C \cap Tor_A$ is finite.*

Both Mordell conjecture and Manin-Mumford conjecture are special cases of the Mordell-Lang conjecture, put forward by Serge Lang in 1965 [7]. The Mordell-Lang conjecture for curves can be stated as follows:

Mordell-Lang Conjecture *Let C be an irreducible curve of a (semi) abelian variety A defined over a number field k. Let Γ be a finitely generated subgroup of $A(k)$ and Γ' a subgroup of the divisible hull of Γ (i.e. for each $x \in \Gamma'$ there exists a non-zero integer n such that $nx \in \Gamma$). If C is not a translate of a (semi) abelian subvariety of A, then $C(k) \cap \Gamma'$ is finite.*

The general statement of the Mordell-Lang conjecture for varieties was proven by McQuillan in 1995 ([8]) building on the break through result of Faltings [5], on the result of Hindry [6] and using a result of Vojta [12]. For more information about this topic we refer the reader to [10]

Next, along this thread of thought, comes the theme of unlike intersections initiated by Bombieri, Masser and Zannier in [1]. In this setting one replaces the set of "special points" (i.e. Γ') with a set of special subvarieties (i.e. algebraic subgroups of A). In order to state the two most relevant conjectures in this setting we need some definitions.

Definition 2.1 A variety $X \subset A$ is called a *torsion variety* (respectively a *translate*) if it is a finite union of translates of algebraic subgroups of A by torsion points (respectively by points).

Definition 2.2 An irreducible variety $X \subset A$ is called *transverse* (respectively *weak-transverse*) if it is not contained in any proper translate (respectively any proper torsion variety).

The Torsion Anomalous Conjecture, which we state below for the case of a weak-transverse curve, has been open for several years:

Torsion Anomalous Conjecture *Let C be a weak-transverse curve in A. Then the set*

$$C \bigcap \left(\bigcup_{\substack{B \text{ algebraic subgroup} \\ \dim B \leq \dim A - 2}} B \right)$$

is finite.

In the above mentioned seminal paper of Bombieri, Masser and Zannier, there is a proof of the Torsion Anomalous Conjecture for transverse curves in an algebraic torus. Their proof is based on the following two statements: (here $\mathcal{B}_2$ denotes the union of the algebraic subgroups of codimension at least 2)

- The points of $C \cap \mathcal{B}_2$ have bounded height.
- The points of $C \cap \mathcal{B}_2$ have bounded degree.

For if the two above conditions are satisfied then the classical Northcott theorem yields the finiteness of $C \cap \mathcal{B}_2$. A central aspect of their proof is that it is effective. This is relevant to find bounds, and even better effective bounds for the height of points in $C \cap \mathcal{B}_2$.

In the course of their investigations around the Torsion Anomalous Conjecture, Checcoli, Veneziano e Viada proved, in [2] a very interesting bound on the height of points of curves contained in a power of a non-CM elliptic curves which we reproduce below. This result improves drastically on some previous bounds proved by the same authors in [3]. The bound proven in [3] is a consequence of a more classical approximation used in connection with the Torsion Anomalous Conjecture, we refer the reader to [3, Theorem 1.1 and Theorem 1.3], see also [11]. The bound in [2] is obtained by introducing new key elements in the proof. It has to be noted that this better bounds are crucial for practical applications, two instances of which will be presented in the final section. In order to state the theorem we need to recall a few definitions regarding heights. Let E be an elliptic curve given in $\mathbb{P}^2$ by the Weierstrass equation $y^2 = x^3 + Ax + B$ with A, B integral. We let $\hat{h}$ be the Néron-Tate height on E^N determined via the Segre embedding. Given a curve $C \subset E^N$ we denote by $h(C)$ the normalised height of C. Finally we denote by $h_W(\alpha)$ the Weil height of an algebraic number α. The following is a simplified version of the main theorem of [2].

Theorem 2.2 *Let E be a non-CM elliptic curve of $\mathbb{Q}$-rank 1. Let $C \subset E^N$ be an irreducibel curve of genus at least 2. Let $C_1 = 145$ and $c_1 = c_1(E) = 2h_W(A) + 2h_W(B) + 4$ with A and B the coefficients of the Weierstrass form. Then $P \in C(\mathbb{Q})$ has height bounded*

$$\hat{h}(P) \leq 4 \cdot 3^{N-2} N! deg\, C (C_1 h(C)(deg\, C) + 4C_1 c_2 (deg\, C)^2 + 2c_1),$$

moreover if $N = 2$

$$\hat{h}(P) \leq C_1 \cdot h(C) deg\, C + 4C_1 c_1 (deg\, C)^2 + 4c_1.$$

If one specialises to a more particular case better bounds can be achieved:

Corollary 2.3 *Suppose* $(x_1, y_1) \times (x_2, y_2)$ *be the affine coordinates of* $E^2 \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ *with* E *defined over* $\mathbb{Q}$*. Let* C *be the curve given in* E^2 *defined by the additional equation* $p(x_1) = y_2$*, with* $p(X) \in k[X]$ *a non-constant polynomial of degree* n*. Then* C *is irreducible and for* $P \in C(k)$ *we have*

$$\hat{h}(P) \leq 2595(h_W(p) + log n + 4c_1)(2n + 3)^2 + 4c_1$$

where $h_W(p) = h_W(1 : p_0 : \ldots : p_n)$ *is the Weil height of the coefficients of* $p(x)$ *and* $c_1 = 2log(3 + |A| + |B|) + 4$

3 Explicit examples

Consider the elliptic curve E defined by the Weierstrass equation

$$y^2 = x^3 + x - 1.$$

In the cartesian product $E \times E \subset \mathbb{P}^2 \times \mathbb{P}^2$ we use affine coordinates $(x_1,\ y_1)$ (respectively (x_2, y_2) on the first factor (respectively the second factor). Next we consider the $\{C_n\}$ family of curves in $E \times E$ given by the additional equation $x_1^n = y^2$. It turns out that the genus $g(C_n) = 4n + 2$ and C_n is irreducible for all n. As consequence of our main theorem we obtain a sharp bound for the height of the points on $C_n(\mathbb{Q})$. Moreover for n large, the points of $C_n(\mathbb{Q})$ will be integral. The bound on the height are so sharp that one can implement in SAGE an exhaustive search. Thus we obtain that

Theorem 3.1 *For all* $n \geq 1$ *the affine rational points of* C_n *are*

$$C_n(\mathbb{Q}) = \{(1,\ \pm 1) \times (1,\ 1)\}$$

Also the next example regards a family of curves, denoted by $\{\mathcal{D}_n\}$, lying in $E^2 = E \times E$. This time the additional equation defining the family of curves is $\Phi_n(x_1) = y_2$ where $\Phi_n(X)$ is the n-th cyclotomic polynomial. It can be shown that the curves $\mathcal{D}_n$ have increasing genus and are irreducible. Moreover, consider the following non-CM elliptic curves.

$$\begin{aligned}
E_1 &: y^2 = x^3 - 26811x - 7320618,\\
E_2 &: y^2 = x^3 - 675243x - 213578568,\\
E_3 &: y^2 = x^3 - 110038419x + 12067837188462,\\
E_4 &: y^2 = x^3 - 2581990371x - 50433763600098.
\end{aligned}$$

These elliptic curves have $\mathbb{Q}$ rank 1. For this family the characterisation of rational points is as follows:

Theorem 3.2 *For $i = 1, 2, 3, 4$ the curves $\mathcal{D}_n \subseteq E_i \times E_i$, there are no rational points other than the point at infinity. And for the curves $\mathcal{D}_n \subseteq E \times E$ we have the following affine rational points:*

$\mathcal{D}_1(\mathbb{Q}) = \{(2, \pm 3) \times (1, 1)\}$; $\mathcal{D}_2(\mathbb{Q}) = \{(2, \pm 3) \times (2, 3)\}$;

$\mathcal{D}_{3^k}(\mathbb{Q}) = \{(1, \pm 1) \times (2, 3)\}$; $\mathcal{D}_{47^k}(\mathbb{Q}) = \{(1, \pm 1) \times (13, 47)\}$;

$\mathcal{D}_{p^k}(\mathbb{Q}) = \emptyset$, *for $p \neq 3, 47$ and if $p = 2$, $k > 1$;*

$\mathcal{D}_6(\mathbb{Q}) = \{(1, \pm 1) \times (1, 1)\} \cup \{(2, \pm 3) \times (2, 3)\}$;

$\mathcal{D}_n(\mathbb{Q}) = \{(1, \pm 1) \times (1, 1)\}$ *if n has at least two distinct prime factors.*

Many other examples can be produced using the same techniques.

References

[1] E Bombieri, D. Masser and U. Zannier, *Intersecting a curve with algebraic sub- groups of multiplicative groups*, Internat. Math. Res. Notices, no. 20 (1999), 1119-1140.

[2] S. Checcoli, F. Veneziano and E. Viada, *The explicit Mordell Conjecture for families of curves (with an appendix by M. Stoll)*, preprint arXiv:1602.04097

[3] S. Checcoli, F. Veneziano and E. Viada, *On the explicit Torsion Anomalous Conjecture*, Journal: Trans. Amer. Math. Soc. 369(2017), 6465-6491.

[4] G.Faltings, *Endlichkeitssätze für ablsche Varietäten über Zahalkörpern*, Invent. Math, 73(1983),no.3, 349-366.

[5] G.Faltings, *The general case of S. Lang's Conjecture*, Barsotti Symposism in Algebraic Geometry (Abano Terme, 1991)Academic Press,San Diego,CA, 15(1994), 175-182.

[6] M.Hindry, *Autour d'une conjecture de Serge Lang*, Invent. Math, 96(1998), 575-603.

[7] S. Lang, *Division points on curves*, Ann. Mat. Pura Appl. (4) 70, 1965, 229-234.

[8] M. McQuillan, *Division Points on semi-abelian varieties*, Invent. Math.120(1995), no. 1, 143-159.

[9] M. Raynaud, *Sous-variete abelenne et points de torsion*, In Arithmetic and Geometry, vol.1,Birkhäuser (1983), 327-352.

[10] Pavlos Tzermos *The Manin-Mumford conjecture: a brief survey*, Bulletin of the London Mathematical Society Vol. 32, Issue 6 (2000), 641-652,

[11] E. Viada, *Explicit height bounds and the effective Mordell-Lang Conjecture*, Proceeding of the Third Italian Number Theory Meeting, Pisa, September 21-25, 2015, Riv. Mat. Univ. Parma, 7 (2016).

[12] P. Vojta, *Integral points on subvarieties of semiabelian varieties*, Invent. Math 126(1996),no. 1,133-181.

Manoj Gyawali
Department of Mathematics and Physics
Roma Tre University
Email: `manoj.gyawali@ncit.edu.np`