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Conjectural estimates on the Mordell-Weil and the Tate-Shavarevich groups of an abelian variety

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The Mordell-Weil theorem states that the group of rational points $A(K)$ on an Abelian variety A defined over a number field K is finitely generated: $A(K) \simeq A(K)_{\text{tors}} \oplus \mathbb{Z}P_1 \oplus \ldots \oplus \mathbb{Z}P_r$. While there exist results on the torsion part, the free part remains less tractable. Even in the particular case of an elliptic curve, there is no way, in general, to compute the rank r or a set of generators $\{P_i\}_{i=1,\ldots,r}$ of this group.

The proof of the Mordell-Weil theorem involves the Tate-Shafarevich group $Ш(A/K)$ of A/K, which measures the obstruction to the Hasse principle. Even if it is not easy to construct a non trivial element of this group, it is still unknown, in the general case, if it is finite.

For some applications, it would be sufficient to bound the "size" of the invariants of the variety. We explore here how could be bounded

1- the product $|Ш(A/K)| \cdot \mathrm{Reg}(A/K)$ of the order of $Ш(A/K)$ and the canonical regulator,

2- the canonical height $\hat{h}_{\mathcal{L}}(P_i)$ of a well chosen system of generators of the free part of $A(K)$, as well as

3- the order $|Ш(A/K)|$ of the Tate-Shafarevic group.

Our bounds are implied by strong but nowadays classical conjectures. We follow the approach of Manin, who proposed a conditional algorithm for finding a basis for the non-torsion rational points of an elliptic curve over $\mathbb{Q}$. We **e**xtend Manin's method to an Abelian variety of arbitrary dimension, defined over an arbitrary number field. Our bounds are **e**xplicit in all the parameters: the Faltings' height $h = h_{Falt}(A/K)$ (which measures the arithmetic complexity of the variety), the absolute value $\mathcal{F} = |N_{K/\mathbb{Q}}\mathcal{F}_{A/K}|$ of the norm of the conductor (which gives information about the places of bad reduction), the dimension g of A, the Mordell-Weil rank $r = \mathrm{rk}(A(K))$, the degree $d = [K : \mathbb{Q}]$, and the absolute value D_K of the discriminant of K.

In this work,

- with point 1, we refine a conjecture of Hindry (related works in different settings have been done also by Hindry-Pacheco and Griffon), and extend to the general case of A/K,

- with point 2, a conjecture of Lang, for elliptic curves over $\mathbb{Q}$,

- with point 3, a result by Goldfeld and Szpiro, towards their conjecture $|\text{Ш}(E/K)| = O(\mathcal{F}_{E/K}^{1/2+\epsilon})$. Furthermore, we **i**mprove their result in the one dimensional case over the field of rational numbers.

The method is based on the Hasse-Weil conjecture which suppose that the L-series of A has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation at 1, and on the Birch-Swinnerton-Dyer conjecture, which translates analytic information into geometric and arithmetic information. We suppose that $|\text{Ш}(A/K)|$ is finite, and conclude with Minskowski's theorem, since the Néron-Tate pairing relates the regulator to the volume of the fundamental domain of the lattice $A(K)/A(K)_{\mathrm{tors}}$.

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