PROCEEDINGS OF THE
ROMAN NUMBER THEORY ASSOCIATION
Volume 4, *Number* 1, *March* 2019, *pages* 91-100

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Zeros of the derivatives of the Riemann zeta function and Dirichlet L-functions

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1 Introduction

The *Riemann zeta function* $\zeta(s)$ is defined as the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

in the half-plane $\Re(s) > 1$ and it is an analytic function on $\mathbb{C} \setminus \{1\}$. Given a primitive Dirichlet character $\chi \pmod q$, with $q > 1$, the Dirichlet L-function $L(s, \chi)$ is entire and satisfies

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for} \quad \Re(s) > 0.$$

It is well-known that the negative even integers are the so-called *trivial zeros* of the Riemann zeta function, while the set

$$Z := \{\rho \in \mathbb{C} \mid \zeta(s) = 0, \rho \notin -2\mathbb{N}_0\}$$

is the set of all *non-trivial zeros* of $\zeta(s)$. These zeros are non-real and they are all located in the right half-plane $\Re(s) > 0$. The Riemann hypothesis (RH) states that, for any $\rho \in Z$, $\Re(\rho) = \frac{1}{2}$.

For a primitive character χ modulo $q \geq 1$, let $\kappa \in \{0, 1\}$ be determined by $\chi(-1) = (-1)^\kappa$. The set of the *trivial zeros* of $L(s, \chi)$ is $\{-\kappa, -2 - \kappa, -4 - \kappa, \dots\}$, while the set of the *non-trivial zeros* is

$$Z(\chi) := \{\rho \in \mathbb{C} \mid L(\rho, \chi) = 0, \rho \neq -2l - \kappa, \forall l \in \mathbb{N}\}.$$

As for the Riemann zeta function, these non-trivial zeros have positive real part, but they are not necessarily non-real. The Generalized Riemann Hypothesis (GRH) states that

$$\Re(\rho) = \frac{1}{2} \quad \text{for any} \rho \in Z \cup Z(\chi).$$

There is an equivalence for RH in terms of zeros of the first derivative of the Riemann zeta function (cf. [8]).

Theorem 1 (Speiser) *The following statements are equivalent*

1. $\zeta(s) \neq 0$ *in* $0 < \Re(s) < \frac{1}{2}$
2. $\zeta'(s) \neq 0$ *in* $0 < \Re(s) < \frac{1}{2}$.

The result below (see [5]) is a sort of analytic analogue of Speiser's theorem. It basically states that $\zeta(s)$ and its first derivative have almost the same number of zeros in the considered region.

Theorem 2 (Levison and Montgomery) *Let $N^-(T)$ (and respectively $N_1^-(T)$) be the number of zeros of $\zeta(s)$ (resp. $\zeta'(s)$) in $\{\sigma + it \mid 0 < \sigma < 1/2, 0 < t < T\}$, counted with multiplicity. Then, for $T \geq 2$*

$$N^-(T) = N_1^-(T) + O(\log T),$$

where the implied constant is absolute.

Similar results can be proved for Dirichlet L-functions. Let $N^-(T,\chi)$ (and respectively $N_1^-(T,\chi)$) be the number of zeros of $L(s,\chi)$ (resp. $L'(s,\chi)$) in the region $\{\sigma+it \mid 0<\sigma<1/2, |t|<T\}$, counted with multiplicity. Moreover, let

$$m := \min\{n \geq 2 \mid \chi(n) \neq 0\},$$

i.e. m is the smallest prime number that does not divide n. Observe that $m = O(\log T)$. The following result holds ([2]).

Theorem 3 (Akatsuka and Suriajaya) *For* $T \geq 2$

$$N^-(T,\chi) = N_1^-(T,\chi) + O(m^{1/2}\log(qT)),$$

where the implied constant is absolute.

This allows to show a Speiser-type equivalence for GRH (again cf. [2]).

Theorem 4 (Akatsuka and Suriajaya) *Let* $\kappa = 0$ *and* $q \geq 216$*. Then the following statements are equivalent*

(i) $L(s,\chi) \neq 0$ *in* $0 < \Re(s) < \frac{1}{2}$.

(ii) $L'(s,\chi)$ *has a unique zero in* $0 < \Re(s) < \frac{1}{2}$.

Let $\kappa = 1$ *and* $q \geq 23$*. Then the following statements are equivalent*

(i) $L(s,\chi) \neq 0$ *in* $0 < \Re(s) < \frac{1}{2}$.

(ii) $L'(s,\chi)$ *has no zeros in* $0 < \Re(s) < \frac{1}{2}$.

Remark 1 *The unique zero of the derivative for* $\kappa = 0$ *is the zero which corresponds to the trivial zero of* $L(s,\chi)$ *at* $s = 0$.

2 Zeros of derivatives of the Riemann zeta function

As for the Riemann zeta function, *non-trivial zeros* of $\zeta^{(k)}(s)$ are non-real zeros. As an upper bound for the real part of the zeros ρ of $\zeta^{(k)}(s)$ one can consider $\Re(\rho) \leq \frac{7}{4}k + 2$, proved by Spira [9], even though this bound can be slightly improved.

Remark 2 *It is interesting to observe the distribution of non-trivial zeros of $\zeta(s)$, $\zeta'(s)$ and $\zeta''(s)$ (cf. [9, Fig. 1]). So far, all non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$, while those of $\zeta'(s)$ and $\zeta''(s)$ move further and further to the right. Moreover, except for a pair of exceptional zeros of $\zeta''(s)$ in the left half-plane, the non-trivial zeros of the first and second derivative seem to appear always in pairs.*

Let now $N(T)$ (resp. $N_k(T)$) be the number of non-trivial zeros ρ of $\zeta(s)$ (resp. $\zeta^{(k)}(s)$), with $0 < \Im(\rho) < T$, counted with multiplicity. Then, von Mangoldt [12] and Berndt [3] respectively proved

$$N(T) = g(T) + O(\log T)$$
$$N_k(T) = h(T) + O(\log T)$$

where

$$g(T) := \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} \quad \text{and} \quad h(T) := \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi}.$$

Under the Riemann hypothesis, the error terms can be improved to

$$O\left(\frac{\log T}{\log\log T}\right) \quad \text{and} \quad O\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$$

respectively. The result for $\zeta(s)$ is due to Littlewood [6], for the first derivative to Akatsuka [1] and the extension to all $k \geq 2$ to Suriajaya [10]. It can be observed that the main term does not depend on k.
Assuming RH, Ge [4] showed that the error term can be improved to

$O\left(\frac{\log T}{\log\log T}\right)$ for the first derivative, while the same result for $k \geq 2$ is expected to hold but it is not proved.

Let now $\sum^{(k)}$ denote the sum over non-trivial zeros ρ of $\zeta^{(k)}(s)$, for $k \geq 0$, with $0 < \Im(\rho) < T$, counted with multiplicity and let

$$f_k(T) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{T}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right) - k\int_2^{\frac{T}{2\pi}}\frac{dt}{\log t}.$$

Since the zeros of $\zeta(s)$ are symmetric with respect to the critical line $\Re(s) = \frac{1}{2}$, one gets

$$\Sigma^{(0)}\left(\Re(s) - \frac{1}{2}\right) = 0.$$

On the other hand, for higher derivatives the zeros are no more symmetric. In [5], Levinson and Montgomery proved that

$$\Sigma^{(k)}\left(\Re(s) - \frac{1}{2}\right) = f_k(T) + O(\log T).$$

Under RH, the error term can be improved to $O((\log\log T)^2)$. This result is due to Akatsuka [1] for $k = 1$ and to Suriajaya [10] for $k \geq 2$.

3 Zeros of derivatives of Dirichlet L-functions

In [13], Yıldırım described a zero-free region for the derivatives of the Dirichlet L-functions.

Theorem 5 (Yıldırım) *For any $\epsilon > 0$, there exists a constant $K = K_{\epsilon,k}$ such that $L^{(k)}(s,\chi) \neq 0$ holds in*

$$\left\{\sigma + it \in \mathbb{C} \,\middle|\, \sigma > 1 + \frac{m}{2}\left(1 + \sqrt{1 + \frac{4k^2}{m\log m}}\right)\right\}$$
$$\cup \{\sigma + it \in \mathbb{C} \,|\, |\sigma + it| > q^K, \sigma < -\epsilon, |t| > \epsilon\}.$$

He also classified the zeros of $L^{(k)}(s,\chi)$ in the following way:

- *trivial zeros*, located in $\{\sigma+it|\sigma\le -q^K, |t|\le\epsilon\}$.
- *vagrant zeros*, located in $\{\sigma+it||\sigma+it|\le q^K, \sigma\le -\epsilon\}$.
- *non-trivial zeros*, located in

$$\left\{\sigma+it\,\middle|\,-\epsilon<\sigma\le 1+\frac{m}{2}\left(1+\sqrt{1+\frac{4k^2}{m\log m}}\right)\right\}.$$

Let now $N_k(T,\chi)$ be the number of non-trivial and vagrant zeros ρ of $L^{(k)}(s,\chi)$, with $|\Im(\rho)|\le T$, counted with multiplicity.

Theorem 6 (Yıldırım) *For $T\ge 2$, we have*

$$N_k(T,\chi)=h(T,\chi)+O(q^K\log T),$$

where

$$h(T,\chi):=\frac{T}{\pi}\log\frac{qT}{2m\pi}-\frac{T}{\pi}.$$

Remark 3 *In this case, the error term is big in terms of the modulus q of the character χ, since K is big. Assuming GRH does not help to improve the error term in terms of q.*

4 Zeros of the first derivative $L'(s,\chi)$

In [2], Akatsuka and Suriajaya proved that there exist no vagrant zeros for the first derivative of a Dirichlet L-function. A zero-free region is described in the result below.

Theorem 7 (Akatsuka and Suriajaya) *Let χ be a primitive Dirichlet character modulo $q>1$. Then $L'(s,\chi)$ has no zeros in*

$$\left\{\sigma+it\,\middle|\,\sigma\le 0, |t|\ge\frac{6}{\log q}\right\}\cup\left\{\sigma+it\,\middle|\,\sigma\le -q^2, |t|\ge\frac{12}{\log|\sigma|}\right\}.$$

Remark 4 *The zero-free region can be extended to the line $\Re(s) = \frac{1}{2}$ under GRH, avoiding zeros of $L(s, \chi)$.*

Remark 5 *Except for a finite number of zeros, each zero of $L'(s, \chi)$ in $\Re(s) \leq 0$ corresponds to a trivial zero of $L(s, \chi)$.*

More precisely, the following result holds.

Theorem 8 (Akatsuka and Suriajaya) *For each $j \in \mathbb{N}_0$:*

- *$L'(s, \chi)$ has exactly a unique zero at*

$$-2j - \kappa + O\left(\frac{1}{\log(jq)}\right)$$

  in the strip $-2j - \kappa - 1 < \Re(s) < -2j - \kappa + 1$.

- *$L'(s, \chi)$ has no zeros on $\Re(s) = -2j - \kappa + 1$.*

1. *If $\kappa = 0$ and $q \geq 7$, then $L'(s, \chi)$ has no zeros in the strip $-1 \leq \Re(s) \leq 0$.*
2. *If $\kappa = 1$ and $q \geq 23$, then $L'(s, \chi)$ has a unique zero in the strip $-2 \leq \Re(s) \leq 0$*

Remark 6 *If the character is odd, the unique zero of $L'(s, \chi)$ corresponds to the trivial zero of $L(s, \chi)$ at $s = -1$.*

For the excluded characters, there is at most a finite number of zeros of $L'(s, \chi)$ in $-1 \leq \Re(s) \leq 0$ if the character is even and in $-2 \leq \Re(s) \leq 0$ if the character is odd. Then, except for a finite number of Dirichlet character, there is a one-to-one correspondence between the zeros of $L'(s, \chi)$ in $\Re(s) \leq 0$ and the trivial zeros of $L(s, \chi)$. Thus, the zeros in the left half-plane of $L'(s, \chi)$ can all be classified as trivial.

One can now focus on the non-trivial zeros in the right half-plane. In [7], Selberg proved that

$$N(T, \chi) = g(T, q) + O(\log(qT)),$$

where $N(T,\chi)$ is the number of zeros ρ of $L(s,\chi)$ with $\Re(\rho) > 0$ and $|\Im(\rho)| \leq T$, counted with multiplicity and

$$g(T,q) := \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi}.$$

He also improved the error term to $O\left(\frac{\log(qT)}{\log\log(qT)}\right)$ under GRH.

In the unconditional case, Akatsuka and Suriajaya [2] improved the error term to $O(m^{1/2}\log(qT))$ for the number of non-trivial zeros of $L'(s,\chi)$ in the right half-plane. Recalling that $m = O(\log q)$, notice that the error term is small.

Assuming GRH, Suriajaya [11] got an error term of the form

$$O\left(\log q + A(q,T)\frac{m^{1/2}\log(qT)}{\log\log(qT)}\right),$$

where $A(q,T)$ is a comparison factor

$$A(q,T) := \min\left\{(\log\log(qT))^{1/2}, 1 + \frac{m^{1/2}}{\log\log(qT)}\right\}.$$

Another improvement to the error term, under GRH, was proved by Ge (2018). He got

$$O\left(\frac{\log(qT)}{\log\log(qT)} + \sqrt{m\log(2m)\log(qT)}\right).$$

Finally, as in the case of $\zeta(s)$ and its derivatives, one can consider the real part distribution of the zeros. Let $\sum^{(0)}$ and $\sum'$ denote the sum over the zeros ρ, with $\Re(\rho) > 0$ and $|\Im(\rho)| \leq T$, counted with multiplicity, of $L(s,\chi)$ and $L'(s,\chi)$ respectively. Then,

$$\Sigma^{(0)}\left(\Re(\rho) - \frac{1}{2}\right) = 0$$

and

$$\Sigma'\left(\Re(\rho) - \frac{1}{2}\right) = f_1(T,\chi) + O(m^{1/2}\log(qT)),$$

where

$$f_1(T,\chi) = \frac{T}{\pi}\log\log\frac{qT}{2\pi} + \frac{T}{\pi}\left(\frac{1}{2}\log m - \log\log m\right) - \frac{2}{q}\int_2^{\frac{qT}{2\pi}} \frac{dt}{\log t}.$$

This result was proved by Akatsuka and Suriajaya [2], while in [11] Suriajaya also proved that, under the generalized Riemann hypothesis, the error term can be improved to

$$O(m^{1/2}(\log\log(qT))^2 + m\log\log(qT) + m^{1/2}\log q).$$

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