PROCEEDINGS OF THE
ROMAN NUMBER THEORY ASSOCIATION
Volume 4, *Number* 1, *March* 2019, *pages* 43-47

Christophe Ritzenthaler

Primes of bad reduction of CM curves of genus 3

Written by Angelo Iadarola

1 Introduction

Let K be a discrete local field with valuation v, $\mathcal{O}$ its valuation ring, $k = \mathcal{O}/\langle\pi\rangle$ its residue field, which we assume to have characteristic different from $2, 3, 5, 7$. In the following, we will be allowed to freely work on finite extensions of K, and we will say "after a finite extension of K". When F is an integral polynomial describing a plane curve, we denote by $\bar{F}$ its reduction modulo π.

Given a genus 3 non-hyperelliptic curve C/K, we want to determine the reduction type of its stable model $\mathcal{C}/\mathcal{O}$, possibly after a finite extension of K. In other words, we want to distinguish between

- C has *hyperelliptic reduction* if the reduction of its stable model $\mathcal{C} \otimes k$ is a hyperelliptic curve of genus 3.
- C has *non-hyperelliptic reduction* if the reduction of its stable model $\mathcal{C} \otimes k$ is a non-hyperelliptic curve of genus 3.
- C has *bad reduction* if the reduction of its stable model $\mathcal{C} \otimes k$ is not a (smooth) curve of genus 3.

Example 1.1 *Let's consider the Klein quartic C over $\mathbb{Q}$, defined by the equation $x^3y + y^3z + z^3x = 0$. It is a smooth plane quartic (non-hyperelliptic) of genus 3. Its reduction is non-singular for every prime different from 7, in which case it is singular, irreducible and has 3 points.*
By changing coordinates, we can write the curve C_1, defined over $\mathbb{Q}(\sqrt{-7})$ and isomorphic to C, given by the equation

$$(x^2 + y^2 + z^2)^2 + \frac{\sqrt{-7} + 7}{2}(x^2y^2 + y^2z^2 + z^2x^2) = 0. \qquad (1)$$

When reducing modulo 7, we get the equation

$$(x^2 + y^2 + z^2)^2 = 0$$

which represents an hyperelliptic curve.

More generally, the main result concerning the relations between plane quartics and hyperelliptic genus 3 curves is the following, which can be found in [1].

Proposition 1.2 *Let $s > 0$ be an integer, $G \in \mathcal{O}[x, y, z]$ a primitive quartic form and $Q \in \mathcal{O}[x, y, z]$ a primitive quadratic form. Assume that $\bar{Q}$ is irreducible and $\bar{Q} = 0$ intersects $\bar{G} = 0$ transversely in 8 distinct $\bar{k}$-points.*
Then the smooth quartic $C/K : Q^2 + \pi^{2s}G = 0$ has hyperelliptic reduction.

For the sake of brevity, we can gather the hypothesis of the previous statement in the following definition.

Definition 1.3 *Given a smooth plane quartic C/K, if we can find a new curve K-isomorphic to C satisfying the hypothesis of the Proposition, we say that C admits a good toggle model.*

By explicit calculation, we can see that, indeed, the Klein quartic admits a good toggle model given precisely by (1), taking $Q = x^2 + y^2 + z^2$, $\pi = \frac{\sqrt{-7}+7}{2}$, $G = x^2y^2 + y^2z^2 + z^2x^2$ and $s = \frac{1}{2}$.

A first new result by the speaker and his coauthors is proving that the converse Proposition 1.2 holds [3, Theorem 2.8, 2.9].

Theorem 1.4 (Lercier, Liu, Lorenzo García, R.) *Let C/K be a smooth plane quartic. Then C has hyperelliptic reduction if and only if C admits a good toggle model over K.*

Further new results are characterizations for having (non-)hyperelliptic reduction, based on a set of invariants of the curve, the Dixmier invariants, which we will discuss in the following section.

2 Dixmier invariants and further results

Let's start by fixing the notation that we will use throughout this section. Given an n-tuple $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_{>0}^{n+1}$, we denote by $\mathbb{P}^{\underline{d}}(K)$ the n-dimensional weighted projective space with weights given by the vector $\underline{d}$. Given a point $\underline{x} = (x_0, \ldots, x_n) \in \mathbb{P}^{\underline{d}}(K)$, possibly after a finite extension of K, we can always find a representative in $\mathbb{P}^{\underline{d}}(\mathcal{O})$ such that one of the coordinates has valuation 0; we call such a representative a *minimal representative* and denote it $\underline{x}^{\min}$. A priori, for a given $\underline{x}$, there are several different minimal representatives, but they all differ by the action of a unity, so, component-wise, they have the same valuation which we call the *normalized valuation with respect to* $\underline{x}$ of x_i and denote $v_{\underline{x}}(x_i)$.

In [2] Dixmier found 7 homogeneous polynomial invariants for the equivalence of ternary quartic forms under the action of $SL_3(\mathbb{C})$, which he called I_3, I_6, I_9, I_{12}, I_{15}, I_{18} and I_{27}, the indices being the degree of the polynomials. Moreover, he proved that they form a *homogeneous set of parameters*, from now on just HSOP, which means that all the invariants are equal to 0 for a quartic form in $\mathbb{C}[x_1, x_2, x_3]$ if and only if these 7 are. We call $\underline{DO}$ the 7-tuple made of these invariants and

we define $\underline{DO}(F)$ as the point of the weighted projective space, with weights given by the indices of the invariants, having as coordinates the evaluation of the invariants at the given ternary quartic form F, unless all the invariants are equal to 0, in which case, of course, we do not get a projective point. Finally, we denote $v_{DO}(I_\bullet(F))$ the normalized valuation with respect to $\underline{DO}(F)$.
Naturally, we can generalize this notation to any tuple of polynomial invariants $\underline{I}$ of any length, so we can now state the first result [3, Theorem 3.14]

Theorem 2.1 (Lercier, Liu, Lorenzo García, R.) *Let $\underline{I}$ be a tuple of invariants, C/K a smooth quartic curve defined by the ternary form $F = 0$. If $\underline{I}$ contains a HSOP over K and k, then C has non-hyperelliptic reduction if and only if $v_I(I_{27}(F)) = 0$.*

Finally the speaker and his coauthors found a new set of explicit invariants $\underline{\iota}$ in [3, Proposition 4.6] which let us give the final result [3, Theorem 1.6]

Theorem 2.2 (Lercier, Liu, Lorenzo García, R.) *There exist 2 sets of invariants, $\underline{DO}$, $\underline{\iota}$ such that a smooth quartic curve C/K defined by the ternary form $F = 0$ has hyperelliptic reduction if*

- $v_{DO}(I_3(F)) = 0$,
- $v_{DO}(I_{27}(F)) = 0$,
- $v_\iota(I_3(F)^5 I_{27}(F)) = 0$.

References

[1] C. H. Clemens, *A scrapbook of complex curve theory*. Plenum Press, New York-London, 1980. The University Series in Mathematics.

[2] J. Dixmier, *On the projective invariants of quartic plane curves*. Adv. in Math., 64:279–304, 1987.

[3] R. Lercier, Q. Liu, E. Lorenzo García, C. Ritzenthaler, *Reduction type of smooth quartics*. To appear.

Angelo Iadarola
Laboratoire Paul Painlevé
Université de Lille
Cité Scientifique, Bâtiment M2
59650 Villeneuve-d'Ascq, France.
email: an.iadarola@gmail.com