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Explicit formulae for averages of Goldbach representations

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1 Introduction

Surely the most famous explicit formula in analytic number theory is the one for the second Chebyshev function, which was proposed by Riemann in his memoir and proved in 1895 by von Mangoldt:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

Since then, several explicit formulae were proved for different arithmetic functions or their means. Hence, one may wish to know if a similar formula exists for the function that counts the number of points with prime coordinates in a triangle, that is

$$\sharp\{(p, p') : p + p' \leq N,\ p \text{ and } p' \text{ prime}\} = \sum_{n \leq N} \sum_{\substack{p,\, p' \text{ prime} \\ p+p'=n}} 1.$$

As it is clear by the last way of writing it, this quantity can be interpreted also as the mean of the number of representations of the integers $n \leq N$ as the sum of two primes (the Goldbach representations). As customary, in order to apply analytic methods, instead of counting just primes one is led to consider the sum extended to all the integers weighted with the von Mangoldt function, so obtaining

$$G_0(N) := {\sum_{n\le N}}' R(n), \quad \text{where } R(n) := \sum_{m+m'=n} \Lambda(m)\Lambda(m').$$

The notation means that, if $N \in \mathbb{N}$, $R(N)/2$ must be subtracted from the first sum. By the classical results on the Goldbach problem, it is expected that $R(n) \sim n\mathfrak{S}(n)$, where $\mathfrak{S}(n)$ is the well known singular series; since this has mean 1, it should be true also that $G_0(N) \sim N^2/2$.

2 Some history of the problem

The first step towards an explicit formula was to prove not only the just mentioned asymptotic behaviour of $G_0(N)$, but also to find a second term in addition to the main one, so obtaining the formula

$$G_0(N) = \frac{1}{2}N^2 - 2\sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)} + E(N),$$

where, as a remark, we note that the sum is absolutely convergent. This was achieved, under the Riemann hypothesis, by:

- Fujii [3] in 1991, with $E(n) \ll (N \log N)^{4/3}$;
- Bhowmik & Schlage-Puchta [1] in 2010, with $E(n) \ll N \log^5 N$;
- Languasco & Zaccagnini [5] in 2012, with $E(n) \ll N \log^3 N$.

As an interesting and natural extension, Languasco and Zaccagnini were led to study the function

$$G_k(N) = \frac{1}{\Gamma(k+1)} \sum_{n<N} R(n) \left(1 - \frac{n}{N}\right)^k \quad \text{for } k \ge 0,$$

which is the Cesàro-Riesz mean for the number of representations defined above and which is equal to the previous function for $k = 0$ and $N \notin \mathbb{N}$. In [6], they proved that unconditionally, for $k > 1$, it holds

$$G_k(N) = \frac{N^2}{\Gamma(k+3)} - 2A_k(N) + B_k(N) + O(N),$$

where

$$A_k(N) = \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+k+2)} N^{\rho+1},$$
$$B_k(N) = \sum_{\rho} \sum_{\rho'} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(\rho+\rho'+k+1)} N^{\rho+\rho'}.$$

For $k = 1$, Goldston and Young [4] were able to obtain a similar result under the Riemann hypothesis.

3 A new approach

All the mentioned results were obtained by using the circle method. In their recent work, instead, Brüdern, Kaczorowski and Perelli [2] have dealt with the problem in a different way and have managed to obtain a formula which is fully explicit.

Their first idea consists in rewriting $G_k(N)$ by means of the simple, but technically critical observation that

$$1 - \frac{m+n}{N} = \left(1 - \frac{n}{N-m}\right)\left(1 - \frac{m}{N}\right),$$

so that

$$\begin{aligned} G_k(N) &= \frac{1}{\Gamma(k+1)} \sum_{m<N} \Lambda(m) \left(1 - \frac{m}{N}\right)^k \sum_{n<N-m} \Lambda(n) \left(1 - \frac{n}{N-m}\right)^k \\ &= \frac{1}{(2\pi \mathrm{i})^2} \int_{(2)} \int_{(2)} \frac{\zeta'}{\zeta}(w) \frac{\zeta'}{\zeta}(s) \frac{\Gamma(w)\Gamma(s)}{\Gamma(w+s+k+1)} N^{w+s} \, ds \, dw, \end{aligned}$$

where to obtain the second equality a double Mellin transform has been performed. As usual, one would like to shift the real part of the lines of integration to $-\infty$ and then evaluate the residues; unfortunately, this is possible only up to a certain point, because there are serious problems of convergence associated to the trivial zeros from some value on. To understand the role of this operation and the importance of trying to move the lines as to the left as possible in the complex plane, we remark that, for example, shifting the lines to 0 (in real parts) gives the already mentioned result by Languasco and Zaccagnini [6].

Hence, the three authors show that the s-integral can be shifted from $\Re s = 2$ to $\Re s = -1/2$. In this operation, two functions arise, namely

$$T_N(w) = -\frac{1}{2\pi \mathrm{i}} \int_{(-\frac{1}{2})} \frac{\zeta'}{\zeta}(s) \, \frac{\Gamma(s)}{\Gamma(w+s+1)} \, N^s \, ds,$$

$$Z_N(w) = \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+w+1)} \, N^{\rho},$$

where either the integral and the sum are are absolutely and compactly convergent in $w > 0$, and so they are both holomorphic there. A key fact, which is proved in Proposition 1 and 2 of [2], is that, for $N \geq 4$, these two functions extend to entire functions with controlled growing ratio.

To be more precise, there exists a real number K such that, for any δ with $0 < \delta < 1$ and any $w = u + \mathrm{i}v$ such that $|w + m| > \delta$ for all integers $m \geq 1$, we have

$$T_N(w) \leq K \, \frac{2^{|u|} \log(|w|+2)}{\delta \, |\Gamma(w+1)|},$$

$$Z_N(w) \leq \frac{K}{\delta \, |\Gamma(w+1)|} \cdot \begin{cases} N^{|u|+1} + 2^{|u|} \log(|w|+2) & \text{if } u \in \mathbb{R}, \\ N^{|u|} \log N + 2^{|u|} \log |w| & \text{if } u \leq -3/2. \end{cases}$$

Using this result, one can now shift the w-integration from $\Re w = 2$ to $\Re w = -M$, where M can vary, and even go to infinity. In this way, the following result, containing the announced completely explicit formula, can be reached.

Theorem 1 *Let us define*

$$\Sigma_\Gamma(N,k) = -\sum_{\nu=1}^{\infty} \operatorname*{res}_{w=-\nu} \frac{\zeta'}{\zeta}(w)\Gamma(w)\frac{N^w}{\Gamma(w+k+1)},$$

$$\Sigma_Z(N,k) = -\sum_{\nu=1}^{\infty} \operatorname*{res}_{w=-\nu} \frac{\zeta'}{\zeta}(w)\Gamma(w)Z_N(w+k)N^w,$$

$$\Sigma_T(N,k) = -\sum_{\nu=1}^{\infty} \operatorname*{res}_{w=-\nu} \frac{\zeta'}{\zeta}(w)\Gamma(w)T_N(w+k)N^w.$$

Then, for N integer, $N \geq 4$, and $k > 0$, we have

$$\begin{aligned}
G_k(N) = {} & \frac{N^2}{\Gamma(k+3)} - 2NZ_N(k+1) + \sum_\rho \Gamma(\rho)Z_N(\rho+k)N^\rho \\
& - 2\frac{\zeta'}{\zeta}(0)\frac{N}{\Gamma(k+2)} + 2\frac{\zeta'}{\zeta}(0)Z_N(k) + NT_N(k+1) \\
& + \left(\frac{\zeta'}{\zeta}(0)\right)^2 \frac{1}{\Gamma(k+1)} - \sum_\rho \Gamma(\rho)T_N(\rho+k)N^\rho - \frac{\zeta'}{\zeta}(0)T_N(k) \\
& + N\Sigma_\Gamma(N,k+1) - \Sigma_Z(N,k) - \frac{\zeta'}{\zeta}(0)\Sigma_\Gamma(N,k) + \Sigma_T(N,k),
\end{aligned}$$

where the sums defined above and the ones over nontrivial zeros of $\zeta(s)$ are absolutely convergent.

To conclude, we make some final remarks.

- The series which define $\Sigma_\Gamma(N,k)$ and $\Sigma_T(N,k)$ are actually asymptotic expansions: this means that, when considering the series truncated at $\nu = M \geq 2$, a sharp error term is obtained, which is roughly $O\left(N^{-M-1}\right)$. This does not hold for $\Sigma_Z(N,k)$: for the tail of its defining series one can only get an error term which is $O\left(N^{-k+\varepsilon}\right)$ for every $\varepsilon > 0$, which is actually an overall error term.

- If one restricts to $k \geq 1/2$, one can recover the same main terms as in Languasco and Zaccagnini [6].

- Following this multiplicative approach, which avoids the use of the circle method, even without the main propositions about the analytic continuation an explicit formula can be reached with an overall error term which is $o(1)$.

References

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