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Counting rational points on genus one curves

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We study the density of rational points on genus one curves C by giving uniform upper bounds for the counting function

$$N(C,B) := \sharp\{P \in C(\mathbb{Q}) : H(P) \leq B\},$$

where the height function H is defined as $H(P) := \max\{|x_0|, ..., |x_n|\}$ for $P = [x_0, ..., x_n]$ with $\gcd(x_0, ..., x_n) = 1$. The main tools to study this problem are descent and determinant methods. We proved new results for genus one curves in two important forms: smooth plane cubic curves and complete intersections of two quadrics in $\mathbb{P}^3$.

Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and r=rank(Jac(C)), then for any positive integer m

$$N(C,B) \ll m^r \left(B^{\frac{2}{3m^2}} + m^2\right) \log B.$$

Taking $m = 1 + [\sqrt{\log B}]$ we obtain $N(C,B) \ll (\log B)^{2+r/2}$. This should be compared with the classical non-uniform bound of Néron: $N(C,B) \sim c_F (\log B)^{r/2}$.

For a non-singular quartic curve C in $\mathbb{P}^3$ defined by a complete intersection of two quadric surfaces $Q_1 = 0$ and $Q_2 = 0$, where $Q_1, Q_2 \in \mathbb{Z}[x_0, x_1, x_2, x_3]^{(2)}$. Then C is also of genus one and Jac(C) is an elliptic curve and again we can use descent argument. We obtain similar estimates as in cubic case

$$N(C,B) \ll m^r \left(B^{\frac{1}{2m^2}} + \log B\right) \log B$$

and

$$N(C,B) \ll (\log B)^{2+r/2}.$$

Moreover, we obtain completely uniform bound for genus one curves in $\mathbb{P}^3$ given in diagonal forms:

$$C : \begin{cases} a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0 \\ b_0x_0^2 + b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = 0 \end{cases}$$

This class contains examples of elliptic curves with arbitrary j-invariants. The main result is

$$N(C,B) \ll_\varepsilon B^{1/2-3/392+\varepsilon}.$$

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