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Diophantine approximation problem with 3 prime variables

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We will prove that the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \leq (\max(p_1, p_2, p_3^k))^{\psi(k)+\varepsilon}$$

where

$$\psi(k) = \begin{cases} (3-2k)/(6k) & \text{se } 1 < k \leq 6/5 \\ 1/12 & \text{se } 6/5 < k \leq 2 \\ (3-k)/(6k) & \text{se } 2 < k < 3 \\ 1/24 & \text{se } k = 3 \end{cases}$$

has infinitely many solutions in prime variables p_1, p_2 and p_3 for any given real number ω, with λ_1, λ_2 and λ_3 non-zero real numbers, not all of the same sign and such that λ_1/λ_2 is not rational, and $1 < k \leq 3$ real (see [1]).

It is easy to see that the hypothesis on the sign is natural, if one wants to approximate all real numbers, and the hypothesis on the ratio λ_1/λ_2 is necessary to avoid trivial cases where the inequality can not hold.

The values for ψ depend on suitable bounds for the relevant exponential sums over prime powers. The proof uses a variant of the circle method technique introduced by Davenport & Heilbronn where the integration on a circle is replaced by the integration on the whole real line, split in a major arc (that provides the main term), an intermediate arc, a minor arc and a trivial arc. The contributions of the last three subsets turn out to be small.

In this kind of problems we can not count "exact hits" hence, we need a measure of "proximity" which can be provided in a number of ways, even tough, the crucial property is the rate of vanishing at infinity that must not be too slow.

Theorem is proved on a suitable sequence X_n with limit $+\infty$, related to the convergent of the fraction λ_1/λ_2 exploiting the fact that we know that there exist infinitely many solutions of the inequality

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| < \frac{1}{q^2}.$$

The main tools used to proved the Theorem are suitable estimations of the L^n-norm of the exponential sums over primes and the Harman technique on the minor arc.

References

[1] A. Gambini, A. Languasco, and A. Zaccagnini. A Diophantine approximation problem with two primes and one k-th power of a prime. *Journal of Number Theory*, 188:210—228, 2018.

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