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Correlations of Ramanujan expansions

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The *c*orrelation ("*s*hifted convolution sum") of any $f, g : \mathbf{N} \to \mathbf{C}$ is

$$C_{f,g}(N,a) \stackrel{def}{=} \sum_{n \le N} f(n) g(n+a). \tag{1}$$

The integer $a > 0$ is the *s*hift. Classic heuristic:

$$C_{f,g}(N,a) \sim S_{f,g}(a)N, \quad S_{f,g}(a) \stackrel{def}{=} \sum_{q=1}^{\infty} \widehat{f}(q)\widehat{g}(q)c_q(a), \tag{2}$$

$S_{f,g}$ is the *s*ingular series and $c_q(a) \stackrel{def}{=} \sum_{j \in \mathbf{Z}_q^*} \cos(2\pi j a/q)$ is the *Ramanujan sum*. Now $S_{f,g}$ is a finite sum, from **Vital Remark:** we have **finite Ramanujan coefficients** $\hat{f}, \hat{g}$, by $f' \stackrel{def}{=} f * \mu$, $g' \stackrel{def}{=} g * \mu$ and Möbius inversion

$$\widehat{f}(q) \stackrel{def}{=} \sum_{\substack{d \le N \\ d \equiv 0 \bmod q}} \frac{f'(d)}{d}, \quad \widehat{g}(q) \stackrel{def}{=} \sum_{\substack{d \le N+a \\ d \equiv 0 \bmod q}} \frac{g'(d)}{d}. \tag{3}$$

With Ram Murty we found the **R**amanujan **e**xact **e**xplicit **f**ormula [J.Number Theory 185(2018),16–47] (here $\varphi(q)$ is *E*uler function):

$$C_{f,g}(N,a) = \sum_{q \le N} \frac{\widehat{g}(q)}{\varphi(q)} \sum_{n \le N} f(n) c_q(n) c_q(a). \tag{\textbf{(}Reef\textbf{)}}$$

It is **n**ot for free: under **B**asic **H**ypotheses, needs some conditions. On [JNT,Th.1] we gave 3 equivalent ones. We give now other 4.

As usual $\omega(d) \stackrel{def}{=} |\{p \text{ prime} : p \text{ divides } d\}|$ and the *Eratosthenes Transform* (**E.t.** for short) of $C_{f,g}$ is

$$C'_{f,g}(N,d) \stackrel{def}{=} \sum_{t|d} C_{f,g}(N,t)\mu(d/t).$$

Under **BH** (Th.1 hypotheses), Reef's equivalent to (F.A.E.):

$$(\textit{Delange Hypothesis}) \qquad \sum_{d} 2^{\omega(d)} \left| C'_{f,g}(N,d) \right| / d < \infty$$

$$(\textit{E.t.Reef}) \qquad C'_{f,g}(N,d) = d \sum_{k \le \frac{Q}{d}} \frac{\mu(k)\widehat{g}(dk)}{\varphi(dk)} \sum_{n \le N} f(n) c_{dk}(n)$$

$$\sum_{d>Q} \frac{1}{d} C'_{f,g}(N,d) \sum_{\substack{\ell > Q \\ \ell | d}} c_\ell(a) = 0, \quad \forall a \in \mathbf{N}$$

$$\lim_{T\to\infty} \sum_{\ell \le T} \sum_{\substack{d>T \\ d \equiv 0 \bmod \ell}} \frac{1}{d} C'_{f,g}(N,d) c_\ell(a) = 0, \quad \forall a \in \mathbf{N}$$

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