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New instances of the Mumford–Tate conjecture

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Let A be a simple abelian variety defined over a number field K of dimension g. Let $MT(A)$ be the Mumford–Tate group of A, which is an algebraic reductive group defined over $\mathbb{Q}$. Let G_K be the absolute Galois group of K, ℓ a prime number and $T_\ell(A)$ the ℓ-adic Tate module of A. Let us consider the following ℓ-adic representation:

$$\rho_\ell : G_K \to Aut(T_\ell) \simeq GL_{2g}(\mathbb{Z}_\ell).$$

We define the ℓ-adic monodromy group G_ℓ as the Zariski closure of the image of ρ_ℓ, it is an algebraic group over $\mathbb{Q}_\ell$.

Conjecture 0.1 (Mumford–Tate '66) *For every prime number ℓ we have*

$$G_\ell^\circ \simeq MT(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

Definition 0.2 *An abelian variety A is fully of Lefschetz type if A satisfies conjecture 0.1 and $MT(A)$ is the Lefschetz group,* i.e *the group of symplectic similitudes which commutes with endomorphisms.*

An abelian variety is of type III, in the sense of Albert classification, if $D := End_K(A) \otimes \mathbb{Q}$ is an indefinite quaternion algebra over a totally real field $F := Z(D)$ of degree e over $\mathbb{Q}$. Let us denote $h := \frac{g}{2e}$ the relative dimension of A in the type III case.

Theorem 0.3 (C.-F., 2017) *Let A be a simple abelian variety of type III. Assume that one of the two conditions is satisfied:*

1. $h \in \{2k+1,\ k \in \mathbb{N}\} \setminus \left\{\frac{1}{2}\binom{2^{m+2}}{2^{m+1}},\ m \in \mathbb{N}\right\}$;
2. $Z(D) = \mathbb{Q}$ *and* $h \notin \Sigma$

The A is fully of Lefschetz type.

The reader can find the definition of Σ in [Can17], for instance:

$$\Sigma = \{4, 6, 8, 16, 36, 64, 70, 100, 128, 144, 196, 216, 256, 324, 400, 484\}.$$

Further applications of this theorem 0.3 can be found in the direction of the Algebraic Sato–Tate conjecture stated by Banaszak and Kedlaya in [BK15]. For instance we can give a new list of abelian varieties which are fully of Lefschetz type and such that the twisted Lefschetz group is connected. In that scenario, those abelian varieties satisfy the Algebraic Sato–Tate conjecture.

References

[BK15] G. Banaszak and K. Kedlaya, *An algebraic Sato-Tate group and Sato-Tate conjecture*, Indiana Univ. Math. J. **64** (2015), no. 1, 245 –274. ↑124

[Can17] V. Cantoral-Farfán, *Torsion for abelian varieties of type III*, ArXiv e-prints (2017), available at `1711.04813`. ↑124

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