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Unlikely Intersections in families of abelian varieties

Fabrizio Barroero

Let n be an integer with $n \geq 2$ and let E_λ denote the elliptic curve in the Legendre form defined by $Y^2 = X(X-1)(X-\lambda)$. Masser and Zannier showed that there are at most finitely many complex numbers $\lambda_0 \neq 0, 1$ such that the two points $\left(2, \sqrt{2(2-\lambda_0)}\right)$ and $\left(3, \sqrt{6(3-\lambda_0)}\right)$ both have finite order on the elliptic curve E_{λ_0}. Later Masser and Zannier proved that one can replace 2 and 3 with any two distinct complex numbers ($\neq 0, 1$) or even choose distinct X-coordinates ($\neq \lambda$) defined over an algebraic closure of $\mathbb{C}(\lambda)$.

In his book, Zannier asks if there are finitely many $\lambda_0 \in \mathbb{C}$ such that two independent relations between the points $\left(2, \sqrt{2(2-\lambda_0)}\right)$, $\left(3, \sqrt{6(3-\lambda_0)}\right)$ and $\left(5, \sqrt{20(5-\lambda_0)}\right)$ hold on E_{λ_0}.

In joint work with Laura Capuano we proved that this question has a positive answer, as Zannier expected in view of very general conjectures. We actually showed a more general result, analogous to the one of Masser and Zannier.

Theorem 1 *Let $C \subseteq \mathbb{A}^{2n+1}$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ with coordinate functions $(x_1, y_1, \ldots, x_n, y_n, \lambda)$, λ non-constant, such that, for every $j = 1, \ldots, n$, the points $P_j = (x_j, y_j)$ lie on E_λ and there are no integers $a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, such that $a_1 P_1 + \cdots + a_n P_n = O$*

identically on C. Then there are at most finitely many $\mathbf{c} \in C$ such that the points $P_1(\mathbf{c}), \ldots, P_n(\mathbf{c})$ satisfy two independent relations on $E_{\lambda(\mathbf{c})}$.

In later works we extended the theorem to abelian schemes.

Fix a number field k and a smooth irreducible curve S defined over k. We consider an abelian scheme $\mathcal{A}$ over S of relative dimension $g \geq 2$, also defined over k. This means that for each $s \in S(\mathbb{C})$ we have an abelian variety $\mathcal{A}_s$ of dimension g defined over $k(s)$.

Let C be an irreducible curve in $\mathcal{A}$ also defined over k and not contained in a proper subgroup scheme of $\mathcal{A}$, even after a base extension. A component of a subgroup scheme of $\mathcal{A}$ is either a component of an algebraic subgroup of a fiber or it dominates the base curve S. A subgroup scheme whose irreducible components are all of the latter kind is called flat.

The following theorem follows from joint works with Laura Capuano and a work of Habegger and Pila in the iso-trivial case.

Theorem 2 *Let k and S be as above. Let $\mathcal{A} \to S$ be an abelian scheme and C an irreducible curve in $\mathcal{A}$ not contained in a proper subgroup scheme of $\mathcal{A}$, even after a finite base change. Suppose that $\mathcal{A}$ and C are defined over k. Then, the intersection of C with the union of all flat subgroup schemes of $\mathcal{A}$ of codimension at least 2 is a finite set.*

FABRIZIO BARROERO
DEPARTEMENT MATHEMATIK UND INFORMATIK
UNIVERSITÄT BASEL
SPIEGELGASSE 1
4051 BASEL, SWITZERLAND.
email: fbarroero@gmail.com