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Value-distribution of cubic L-functions

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This is a report of the results obtained in a joint work by Amir Akbary and Alia Hamieh. The study on the distribution of values of L-functions associated with quadratic Dirichlet characters in the half plane $\Re(s) > \frac{1}{2}$ has been investigated by several authors. One of the early results is obtained by Chowla and Erdős in 1953. Let d be an integer such that d is not a perfect square and $d \equiv 0, 1 \pmod 4$. Suppose that, for $\Re(s) > 0$, we have

$$L_d(s) = \sum_{n=1}^{\infty} \frac{\left(\frac{d}{n}\right)}{n^s}.$$

Here the quadratic Dirichlet character of the function $L_d(s)$ is determined by the Kronecker symbol $\left(\frac{d}{\cdot}\right)$. The distribution of values of $L_d(s)$ in the half-line $\sigma > \frac{3}{4}$ for varying d has been described by the authors in [1] as the following theorem.

Theorem 1 (Chowla-Erdős) *If $\sigma > 3/4$, we have*

$$\lim_{x\to\infty} \frac{\#\{0 < d \leq x; d \equiv 0, 1 \pmod 4 \text{ and } L_d(\sigma)\} \leq z\}}{x/2} = G(z),$$

where $G(0) = 0, G(\infty) = 1$ and $G(z)$ is the distribution function, which is a continuous and strictly increasing function of z.

In 1970 Elliott reconsidered this problem for $\sigma = 1$ and extended Chowla-Erdős theorem. The following is proved in [2].

Theorem 2 (Elliott) *There is a distribution function $F(z)$ such that*

$$\frac{\#\{0 < -d \le x; d \equiv 0, 1 \pmod 4) \text{ and } L_d(1) \le z\}}{x/2} = F(z) + O\left(\sqrt{\frac{\log\log x}{\log x}}\right)$$

holds uniformly for all real z, and real $x \ge 9$. $F(z)$ has a probability density, may be differentiated any number of times, and has the characteristic function

$$\varphi_F(y) = \prod_p \left(\frac{1}{p} + \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p}\right)^{-iy} + \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)^{-iy}\right)$$

which belongs to the Lebesgue class $L^1(-\infty, \infty)$.

This theorem provides detailed information on the distribution function in Chowla-Erdős theorem for $\sigma = 1$ with an explicit error term. In 1970 Elliott explored similar expressions for several other functions (see [3, 4, 5]).

In 2015, Mourtada and Murty [6] described the density function M_σ for the values of the logarithmic derivative of $L_d(s)$ for $\sigma > \frac{1}{2}$ in the following theorem.

Theorem 3 (Mourtada-Murty) *Let $\sigma > \frac{1}{2}$ and assume the GRH (the Generalized Riemann Hypothesis for $L_d(s)$). Let $\mathcal{F}(Y)$ denote the set of the fundamental discriminants in the interval $[-Y, Y]$ and let $N(Y) = \#\mathcal{F}(Y)$. Then, there exists a probability density function M_σ, such that*

$$\lim_{Y\to\infty} \frac{1}{N(Y)} \#\{d \in \mathcal{F}(Y); (L_d'/L_d)(\sigma) \le z\} = \int_{-\infty}^{z} M_\sigma(t)dt.$$

Moreover, the characteristic function $\varphi_{F_\sigma}(y)$ of the asymptotic distribution function $F_\sigma(z) = \int_{-\infty}^{z} M_\sigma(t)dt$ is given by

$$\varphi_{F_\sigma}(y) = \prod_p \left(\frac{1}{p+1} + \frac{p}{2(p+1)} \exp\left(-\frac{iy \log p}{p^\sigma - 1} \right) + \frac{p}{2(p+1)} \exp\left(\frac{iy \log p}{p^\sigma + 1} \right) \right).$$

Here Amir Akbary and Alia Hamieh note that it is possible to remove the GRH assumption in Theorem 3 by applying an appropriate zero density theorem for L-functions of quadratic Dirichlet characters. They describe their approach for certain cubic L-functions.

Notice that if d is a fundamental discriminant then

$$L_d(s) = \frac{\zeta_{\mathbb{Q}(\sqrt{d})}(s)}{\zeta(s)}, \tag{1}$$

where $\zeta_{\mathbb{Q}(\sqrt{d})}(s)$ is the Dedekind zeta function of $\mathbb{Q}(\sqrt{d})$ and $\zeta(s)$ is the Riemann zeta function. For $k = \mathbb{Q}(\sqrt{-3})$, let $\mathfrak{O}_k = \mathbb{Z}[\zeta_3]$ be the ring of integers of k, where $\zeta_3 = e^{\frac{2\pi i}{3}}$. Let

$$C := \{c \in \mathfrak{O}_k; c \neq 1 \text{is square free and } c \equiv 1 \pmod{\langle 9 \rangle}\}.$$

Similar to (1), we can define

$$L_c(s) = \frac{\zeta_{k(c^{1/3})}(s)}{\zeta_k(s)}, \tag{2}$$

where $\zeta_{k(c^{1/3})}(s)$ is the Dedekind zeta function of the cubic field $k(c^{1/3})$ for $c \in C$.

We set

$$\mathcal{L}_c(s) = \begin{cases} \log L_c(s) & \text{(Case 1)}, \\ (L_c'/L_c)(s) & \text{(Case 2)}. \end{cases}$$

The following was the main result of this talk.

Theorem 4 (Akbary-Hamieh) *Let $\sigma > \frac{1}{2}$. Let $\mathcal{N}(Y)$ be the the number of elements $c \in C$ with norm not exceeding Y. There exists a smooth density function M_σ such that*

$$\lim_{Y\to\infty} \frac{1}{\mathcal{N}(Y)} \#\{c \in C : N(c) \leq Y and\ \mathcal{L}_c(\sigma) \leq z\} = \int_{-\infty}^{z} M_\sigma(t)dt.$$

The asymptotic distribution function $F_\sigma(z) = \int_{-\infty}^{z} M_\sigma(t)dt$ can be constructed as an infinite convolution over prime ideals $\mathfrak{p}$ of k,

$$F_\sigma(z) = *_{\mathfrak{p}} F_{\sigma,\mathfrak{p}}(z),$$

where

$$F_{\sigma,\mathfrak{p}}(z) = \begin{cases} \dfrac{1}{N(\mathfrak{p})+1}\delta + \frac{1}{3}\left(\dfrac{N(\mathfrak{p})}{N(\mathfrak{p})+1}\right) \sum_{j=0}^{2} \delta_{-a_{\mathfrak{p},j}}(z) & \text{if } \mathfrak{p} \nmid \langle 3 \rangle, \\ \delta_{a_{\mathfrak{p},0}}(z) & \text{if } \mathfrak{p} \nmid \langle 1 - \zeta_3 \rangle. \end{cases}$$

Here $\delta_a := \delta(z-a)$, δ is the Dirac distribution, and

$$a_{\mathfrak{p},j} := a_{\mathfrak{p},j}(\sigma) = \begin{cases} 2\Re\left(\log(1 - \zeta_3^j N(\mathfrak{p})^{-\sigma}\right) & \text{in (Case 1)}, \\ 2\Re\left(\dfrac{\zeta_3^j \log(N(\mathfrak{p}))}{N(\mathfrak{p})^\sigma - \zeta_3^j}\right) & \text{in (Case 2)}. \end{cases}$$

Moreover, the density function M_σ can be constructed as the inverse Fourier transform of the characteristic function $\varphi_{F_\sigma}(y)$, which in (Case 1) is given by

$$\varphi_{F_\sigma}(y) = \exp(-2yi\log(1-3^{-\sigma})) \prod_{\mathfrak{p}\nmid\langle 3\rangle} \left(\frac{1}{N(\mathfrak{p})+1} + \frac{1}{3}\frac{N(\mathfrak{p})}{N(\mathfrak{p})+1} \sum_{j=0}^{2} \exp \left(-2yi \log\left|1 - \frac{\zeta_3^j}{N(\mathfrak{p})^\sigma}\right|\right)\right),$$

and in (Case 2) is given by

$$\varphi_{F_s}(y) = \exp\left(-2yi\Re\frac{\log(3)}{3^\sigma - 1}\right)\prod_{\mathfrak{p}\nmid\langle 3\rangle}\left(\frac{1}{N(\mathfrak{p})+1} + \frac{1}{3}\frac{N(\mathfrak{p})}{N(\mathfrak{p})+1}\sum_{j=0}^{2}\exp\left(-2yi\right.\right.$$
$$\left.\left.\cdot\Re\left(\frac{\zeta_3^j\log(N(\mathfrak{p}))}{N(\mathfrak{p})^\sigma - \zeta_3^j}\right)\right)\right).$$

As an application of the above theorem note that according to the class number formula

$$\mathcal{L}_c(1) = \frac{(2\pi)^2\sqrt{3}h_c R_c}{\sqrt{|D_c|}}$$

The value $\mathcal{L}_c(1)$ has some arithmetic significance. Here, h_c, R_c and $D_c = (-3)^5(N(c))^2$ are respectively the class number, the regulator, and the discriminant of the cubic extension $K_c = k(c^{1/3})$ (see [7], page 427] for more explanation). On the other hand by the Brauer-Siegel theorem we have $\log(h_c R_c) \sim \log|D_c|^{1/2}$, whenever $N(c) \to \infty$ (Note that the number fields K_c all have a fixed degree (namely 6) over $\mathbb{Q}$).

Corollary 5 *Let* $E(c) = \log(h_c R_c) - \log|D|^{1/2}$. *Then*

$$\lim_{Y\to\infty}\frac{1}{\mathcal{N}(Y)}\#\{c \in C : N(c) \le Y and\, E(c) \le z\} = \int_{-\infty}^{z+\log(4\sqrt{3}\pi^2)} M_1(t)dt,$$

where $M_1(t)$ *is the smooth function described in Theorem 4 (Case 1) for* $\sigma = 1$.

As another application note that the Euler-Kronecker constant of a number field K is defined by the relation

$$\gamma_K = \lim_{s\to 1}\left(\frac{\zeta_K'(s)}{\zeta_K} + \frac{1}{s-1}\right).$$

From (2) We concluded that $\dfrac{L_c'(1)}{L_c(1)} = \gamma_{K_c} - \gamma_k$. Thus, we get the following corollary of Theorem 4 (Case 2), since γ_k is fixed.

Corollary 6 *There exists a smooth function $M_1(t)$ (as described in Theorem 4 (Case 2) for $\sigma = 1$) such that*

$$\lim_{Y\to\infty} \frac{1}{\mathcal{N}(Y)} \#\{c \in C : N(c) \leq Y and\, \gamma_{K_c} \leq z\} = \int_{-\infty}^{z-\gamma_k} M_1(t)dt.$$

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